An electronic warfare perspective on time difference of arrival estimation subject to radio receiver imperfections

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Abstract

In order to ensure secure communication in digital military radio systems, multiple methods are used to protect the transmission from being intercepted by enemy electronic warfare systems. An intercepted transmission can be used to estimate several parameters of the transmitted signal such as its origin (position or direction) and of course the transmitted message itself. The methods used in traditional electronic warfare direction-finding systems have in general poor performance against wideband low power signals while the considered correlation-based time-difference of arrival (TDOA) methods show promising results.

The output from a TDOA-based direction-finding system using two spatially separated receivers is the TDOA for the signal between the receiving sensors which uniquely describes a hyperbolic curve and the emitter is located somewhere along this curve. In order to measure a TDOA between two digital radio receivers both receiver systems must have the same time and frequency references to avoid degradation due to reference imperfections. However, in some cases, the receivers are separated up to 1000 km and can not share a common reference. This is solved by using a reference module at each of the receiver sites and high accuracy is achieved using the NAVSTAR-GPS system but, still, small differences between the outputs of the different reference modules occurs which degrades the performance of the system.

In a practical electronic warfare system there is a number of factors that degrade the performance of the system, such as non-ideal antennas, analog receiver filter differences, and the analog to digital converter errors. In this thesis we concentrate on the problems which arises from imperfections in the reference modules, such as time and frequency errors.
The performance of both time- and frequency-domain based TDOA estimators are studied and compared to the Cramér-Rao lower bound. Also, the effects from the reference frequency errors are studied in terms of performance. Any difference between the time reference signals between the two receiver systems produces a biased estimate. This bias is directly linked to the time-difference, or error, between the time reference outputs of the two reference modules. In order to digitize, or sample, the signal it needs to be transposed, or mixed, to near baseband using a superheterodyne receiver controlled by the reference module. A difference in phase between the mixer frequency reference outputs using two reference modules will not affect the performance of the TDOA estimation. However, a receiver oscillator frequency error due to the frequency difference between the frequency reference outputs of two different reference modules will result in a noise-like degradation of the TDOA estimation process. A measure of this degradation is presented as the estimators robustness against a frequency error.
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Chapter 1

Introduction

In order to ensure secure communication in digital military radio systems, multiple methods are used to protect the transmission from being intercepted by enemy electronic warfare systems. An intercepted transmission can be used to estimate several parameters of the transmitted signal such as origin (position or direction), modulation, coding, encryption and of course the transmitted message itself. During World War II, the main method to ensure unintercepted transmissions was to use burst transmissions, such as the 454 ms transmissions used by German Wolfpack submarine groups [1]. Electronic warfare systems with intercept and direction-finding capabilities of these signals followed shortly. Modern methods to avoid interception include both frequency-hopping spread-spectrum and direct-sequence spread-spectrum communication systems operating with low radiated power and directional antennas. Emerging technologies also include filtered spreading codes to further reduce the probability of interception [15]. A typical modern tactical military communication system uses some 100,000 hops per second and more than 10 MHz of instantaneous bandwidth. Future communication systems are expected to operate at higher hop-rates and larger bandwidths.

The methods used in traditional electronic warfare direction-finding systems to intercept modern frequency-hopping systems have in general poor performance against wideband low power signals [11] while correlation-based time-difference of arrival (TDOA) methods show promising results [3]. The considered method is applicable to other wideband signals, such as radar, sonar or acoustic signals, as well.
1.1 Direction-finding and positioning

Traditional direction-finding systems employing the triangulation principle use an antenna array to estimate the direction of arrival by measuring phase-differences between the individual antenna elements. Each direction-finding system then produces a direction-of-arrival estimate for the intercepted signal. An estimated emitter position can be calculated from the intersection of the individual directions of arrival as illustrated in Figure 1.1. In this thesis, two dimensional direction-finding and positioning is considered.

Existing phase-measuring direction-finding systems are often designed to perform well against frequency-hopping systems using narrowband direction-finding methods such as MUSIC or Watson-Watt which gives poor direction-finding performance when applied to wideband low power signals [11]. To meet the need for direction-finding and positioning of wideband low power signals, a correlation-based TDOA method using two or more, spatially separated receivers, is considered. The output from a TDOA-based direction-finding system is the TDOA for the signal between a pair of receiving sensors which uniquely describes a hyperbolic curve and the
1.1. Direction-finding and positioning

Figure 1.2: Two TDOAs between three sensors gives three hyperbolic curves which, here, uniquely intersects at the emitter position.

The emitter is located somewhere along this curve. More than two receiving sensors gives three, or more, hyperbolic curves which intersect at a position describing the emitter position as illustrated in Figure 1.2. However, in a practical scenario both bias and variance of the estimated TDOAs will affect the hyperbolic curves which will not intersect in a point but rather describe an area from which the emitter position is calculated. How to calculate the emitter position from this area is a problem with several proposed solutions [7]. If the sensor positions are selected without consideration to the emitter position the hyperbolic curves may intersect at more than one position. The problem of selecting suitable sensor positions to avoid more than one intersection between the hyperbolic curves is not considered here. In this thesis the focus is on estimating the TDOAs which are used to calculate the hyperbolic curves.
The TDOA between two sensors separated by a distance \( d \) gives a hyperbolic curve. For large distances this hyperbolic curve can be approximated by a straight line giving an approximate (asymptotic) direction of arrival.

When the distance \( d \) between the sensors is short relative the distance between the emitter and the sensors, an approximate direction of arrival \( \alpha \) can be calculated as

\[
\alpha = \sin^{-1} \left( \frac{v \Delta t}{d} \right).
\]  

(1.1)

The propagation speed of the signal is denoted \( v \) and \( \Delta t \) is the TDOA between the receiving sensors. An illustration of this approximation is given in Figure 1.3 where the fundamental ambiguity is omitted.

To summarize, the TDOA for a signal between two receiving sensors gives a hyperbolic curve which describes the possible locations of the emitter. Three, or more, sensors will produce three, or more hyperbolic curves and the position of the emitter is given by the intersection between these curves.
1.2 A basic system model and two estimators

In the previous chapter it is shown that the TDOA of the signal of interest between two sensors gives a hyperbolic curve and an approximate direction to the emitter. In the following, a basic correlation-based model is derived to show how the TDOA can be calculated using the outputs of two generic sensors \( r_1(t) \) and \( r_2(t) \). Throughout this thesis a complex-valued signal model is used and here the signal of interest is modelled as a zero-mean complex-valued wide sense stationary baseband process denoted \( s(t) \). The signal is thus characterized by its auto-correlation function

\[
\phi_s(\tau) = \mathbb{E}\{s(t+\tau) s^*(t)\}.
\]  

Under the assumption that \( s(t) \) is wide sense stationary the auto-correlation function is independent of time \( t \) and a function of \( \tau \) only. Furthermore, it is assumed that \( s(t) \) is strictly bandlimited into the frequency range \((-W,W) \) Hz and that its power spectral density is continuous in frequency. That is, the transmitted signal is assumed to be strictly band-limited, but broadband. In the basic model, the signal \( s(t) \) is received at two spatially separated sensors with additive noise. The time of arrival of the signal will differ between the two sensors resulting in a TDOA denoted \( \Delta_t \). The TDOA is independent of time for non-moving emitter and receiving sensors. Now, the outputs of the two receiving sensors with ideal non-dispersive complex-valued additive white Gaussian noise (AWGN) are given by

\[
r_1(t) = s(t) + z_1(t)
\]  

and

\[
r_2(t) = s(t - \Delta_t) + z_2(t).
\]  

The noises \( z_1(t) \) and \( z_2(t) \) are assumed zero-mean, mutually uncorrelated and uncorrelated to \( s(t) \). The cross-correlation function (CCF) between the two received signals \( r_1(t) \) and \( r_2(t) \) is defined as

\[
\mathbb{E}\{r_1(t+\tau) r_2^*(t)\}
\]  

which is evaluated using (1.3)-(1.4)

\[
\mathbb{E}\{r_1(t+\tau) r_2^*(t)\} = \mathbb{E}\{s(t+\tau) s^*(t - \Delta_t)\} + \mathbb{E}\{s(t+\tau) z_2^*(t)\} + \mathbb{E}\{z_1(t+\tau) s^*(t - \Delta_t)\} + \mathbb{E}\{z_1(t+\tau) z_2^*(t)\}
\]  

(1.6)
where the only non-zero term is the first signal-signal component since the noises are assumed both mutually uncorrelated and uncorrelated to \( s(t) \). The CCF is then a function of \( \tau \) only since \( s(t) \) is assumed wide sense stationary

\[
\phi(\tau) \triangleq E\{r_1(t+\tau)r_2^*(t)\} = E\{s(t+\tau)s^*(t-\Delta_t)\} = \phi_s(\tau + \Delta_t) \tag{1.7}
\]

where the second equality follows from (1.6) and the third from (1.2).

Considering an electronic warfare scenario there are typically several time and/or frequency overlapping signals in the received sequences. If the signals only are time overlapping, simple frequency filtering is achieved by employing a frequency-domain correlation-based method. Separation of time and frequency overlapping signals can be done using spatial filtering [9]. Based on the idealized basic model described above, two ways of estimating the TDOA are studied – time-domain and frequency-domain based estimators. The time-domain TDOA estimator is based on the relation in (1.7), that is finding the argument that maximizes the amplitude of the CCF

\[
\Delta_t = -\arg\max_{\tau} |\phi(\tau)|. \tag{1.8}
\]

The time-domain TDOA estimator is then given by the lag \( \tau \) that maximizes the amplitude of the estimated CCF

\[
\tilde{\Delta}_t = -\arg\max_{\tau} |\tilde{\phi}(\tau)|. \tag{1.9}
\]

In Figure 1.4, the CCF between the received signals from two sensors is calculated for a white Gaussian noise signal. The peak at \(-1 \mu s\) corresponds to a TDOA of \( \Delta_t = 1 \mu s \).

The considered frequency-domain TDOA estimator is based on the cross spectral density \( \Phi(f) \), and in particular the phase slope of the cross spectral density which is used to calculate the TDOA. The cross spectral density is defined as the Fourier transform of the CCF [12]

\[
\Phi(f) \triangleq \mathcal{F}\{\phi(\tau)\} = \Phi_s(f)e^{j2\pi f\Delta_t} \tag{1.10}
\]

where \( \Phi_s(f) = \mathcal{F}\{\phi_s(\tau)\} \) is the power spectral density of \( s(t) \). In (1.10), the second equality follows from (1.7) and the time-shift property of the Fourier transform. The phase of the cross spectral density \( \Gamma(f) \triangleq \angle\Phi(f) \) is using (1.10) given by

\[
\Gamma(f) = 2\pi f\Delta_t \tag{1.11}
\]
1.2. A basic system model and two estimators

Figure 1.4: The CCF is calculated for an example signal in noise. Here, the true TDOA $\Delta_t = 1 \mu s$ corresponds to the position of the peak at $-1 \mu s$ in accordance with (1.9).

since the phase of the real-valued power spectral density is zero. The phase of the cross spectral density in (1.11) is linear in $f$ with a slope determined by the TDOA $\Delta_t$ as illustrated in Figure 1.5 for a typical bandlimited signal. Within the central high SNR region the phase is linear while the low SNR region is dominated by the noise resulting in a random, non-linear phase.

The electronic warfare scenario assumes no a priori information of the signal and an estimator needs to perform well under such circumstances. The considered frequency-domain based TDOA estimator is a least-squares estimator which requires no prior knowledge of the signal or the noise. The main idea is to fit a straight line to the phase curve of the estimated cross spectral density [18],[26]. In Chapter 3 the estimators are analyzed in detail.
In this thesis the focus is on modelling a practical TDOA direction-finding system and the performance degradation of a correlation-based TDOA estimator using imperfect receiver systems. In the considered electronic warfare scenario the SNR is assumed low which results in TDOA estimates with high variance. In order to reduce this variance block-averaging is used. That is, several spectral estimates calculated from independent data records are averaged and a TDOA estimate with reduced variance is obtained. In the following, the effects of the imperfect receiver systems are analyzed in terms of bias and variance of the TDOA estimates.

In previous work, especially [14] and [18], the effects of a limited acquisition interval are analyzed briefly for a perfect receiver system. However, the effects of imperfect receiver systems in combination with a limited acquisition interval and block-averaging have not been published in the literature. In this thesis, the main contributions are the models of the
1.3. Contributions and outline

imperfect receiver systems and how these imperfections affect the TDOA estimation process. In particular the effects of block-averaging in combination with a receiver oscillator frequency error and a limited acquisition interval are considered from a practical viewpoint.

The following Chapters are organized as

- Chapter 2 - System models,
  describing the considered system models including the effects of limited acquisition intervals with receiver and reference imperfections. In particular the case of receiver frequency error in combination with a limited acquisition interval is considered.

- Chapter 3 - Time- and frequency-domain TDOA estimation,
  the system models derived in Chapter 2 are used in estimating the TDOA. The effects of different errors on the proposed TDOA estimators are studied and in particular the effects of timing errors and frequency errors are studied. The estimator performance is compared to the Cramér-Rao lower bound.

- Chapter 4 - Conclusions and topics for future research,
  a summary of the contributions and conclusions to be drawn from the results. Also, some topics for further research are presented.

The work in this thesis is in part based on the following publications


1.4 Notation

- $x(t), y(t)$: Time continuous signal
- $x[n], y[n]$: Time discrete signal
- $x^*$: Complex conjugate of $x$
- $p_N[n], p_M^*[m]$: Rectangular windows of length $N$ and $M$, respectively
- $f_s$: Sampling frequency (Hz)
- $\Delta_t$: TDOA in continuous time (seconds)
- $\Delta$: TDOA in discrete time $\Delta = \Delta_t f_s$ (samples)
- $\gamma$: Receiver oscillator phase difference (radians)
- $\epsilon$: Receiver frequency difference relative $f_s$
- $\mu_x$: Normalized frequency for signal $x$, $\mu_x = f_x / f_s$
- $\varphi$: Receiver oscillator phase (radians)
- $\hat{\theta}$: Estimate of the parameter $\theta$
- $\phi_x(\tau), \phi_x[m]$: Auto-correlation function of the signal $x$
- $\phi(\tau), \phi[m]$: Cross-correlation function (CCF) between two signals
- $\Phi_x(f), \Phi_x[k]$: Power spectral density of the signal $x$
- $\Phi(f), \Phi[k]$: Cross spectral density between two signals
- $\Gamma(f), \Gamma[k]$: Phase of the cross spectral density
- $T$: Acquisition interval in seconds
- $N$: Acquisition interval in samples $N = T f_s$
- $L$: Length of block used in block-averaging (samples)
- $B$: Number of blocks used for non-overlapping block-averaging $N = LB$
- $M$: Length of CCF, $M = 2N - 1$ (samples)
- $2W$: Bandwidth of the complex-valued signal (Hz)
- $U$: Number of sources in a multisource scenario
- $F_M\{x[n]\}$: The discrete Fourier transform (DFT) of $x[n]$ of length $M$ for $k = 1 - N, ..., N - 1$
- $E\{x[n]\}$: Expected value of the random variable $x[n]$
- $\angle x[n]$: Phase of the complex-valued variable $x[n]$
- $\text{sinc}(x)$: $\sin(\pi x) / \pi x$
Chapter 2
System models

In the considered electronic warfare scenario the signal of interest is unknown and received at low SNR. Most modern electronic warfare systems consist of digital radio receivers with digital signal processing to analyze the acquired signals [22]. To model a practical electronic warfare system with digital receivers, a simple but yet detailed model using superheterodyne receivers is derived. In order to keep the digital receivers near time- and frequency-synchronized, reference modules are used at each of the receiver sites. In the following it is shown how imperfections in, or more accurately, differences between the reference modules will affect the models of the considered correlation-based TDOA direction-finding system. In previous work baseband models are often considered for simplicity [14],[18]. However, to properly model the receiver system imperfections such as the receiver frequency error, the limited acquisition interval and the receiver mixing in a practical system need to be considered.

2.1 Analog to digital conversion of the received signals

A typical modern electronic warfare system uses digital receivers and the signal processing is performed using sampled data. The received data are digitized using the sampling frequency $f_s$ Hz and $K$ bits. In the considered electronic warfare scenario the signal of interest is assumed to have low SNR, that is the channel noise power will be in the same order as the signal power. In a practical receiver system the analog-to-digital converter (ADC) input levels are adjusted so that the full scale
amplitude never is reached. This leads to a fewer number of effective bits than the maximum $K$ bits. However, in [6] it is shown that the effects of a reduced number of effective bits can be suppressed by using a larger acquisition interval. Thus, in this thesis ideal sampling is assumed, that is no amplitude quantization effects are included in the analysis since the effects of non-ideal sampling can be reduced by a larger acquisition interval.

In modern digital receivers the analog filters have high out-of-band attenuation, that is the effects of aliasing are negligible and is not considered in this thesis.

2.2 Fractional delays of sampled data

In this thesis, the notation $s[n - \Delta]$ and $\phi_s[m + \Delta]$ are commonly used to describe delayed versions of discrete time sequences. The TDOA $\Delta$ between the two sensors is not an integer in a practical system and the delayed sequences are thus formally undefined. However, the fractionally delayed sequences are only used to analyze the received sequences and are defined as follows. Considering a noise free scenario, the digitized versions of the received signals in (1.3)-(1.4) are given by

$$r_1[n] = s(t) |_{t=nf_s^{-1}} \triangleq s[n]$$  \hspace{1cm} (2.1)

and

$$r_2[n] = s(t - \Delta t) |_{t=nf_s^{-1}} \triangleq s[n - \Delta]$$  \hspace{1cm} (2.2)

with $\Delta = \Delta_t f_s$. The CCF between (2.1)-(2.2) is then given by

$$\phi[m] = \mathbb{E} \{ r_1[n + m] r_2^*[n] \}$$  \hspace{1cm} (2.3)

which using (2.1)-(2.2) is evaluated to

$$\phi[m] = \mathbb{E} \{ s[n + m] f_s^{-1} \} s^*[n - \Delta f_s^{-1}]$$  \hspace{1cm} (2.4)

By defining the delayed auto-correlation function of $s[n]$ as

$$\phi_s[(m + \Delta) f_s^{-1}] \triangleq \phi_s[m + \Delta] = \mathbb{E} \{ s[n + m] s^*[n - \Delta] \}$$  \hspace{1cm} (2.5)

the time-discrete CCF in (2.4) can be written on the form

$$\phi[m] = \phi_s[m + \Delta].$$  \hspace{1cm} (2.6)
Now, both \( s[n - \Delta] \) and \( \phi_s[m + \Delta] \) are defined for non-integer \( \Delta \)s and the analysis is valid for all bandlimited signals \( s[n] \). Note that it is only in the analysis of the considered methods that the fractionally delayed signals are used, not in the method itself.

### 2.3 Limited acquisition intervals

In a practical data acquisition system the acquisition interval is limited to \( T \) seconds. This is not always considered, in theoretical models used to analyze the behavior and performance of TDOA direction-finding systems. However, in presence of receiver imperfections the limited acquisition interval needs to be considered. In this thesis, the analysis is made using a data acquisition interval of \( T \) seconds or \( N = Tf_s \) samples. That is, the considered stochastic processes are defined for an infinite time interval but only \( T \) seconds, or \( N \) samples, are used in the analysis. The selection of the \( N \) samples is made using a deterministic asymmetric rectangular window \( p_N[n] \) described by

\[
p_N[n] = \begin{cases} 
1 & -\frac{N}{2} + 1 \leq n \leq \frac{N}{2} \\
0 & \text{otherwise}
\end{cases}
\]  

(2.7)

where for simplicity, \( N \) is always assumed to be an even integer. Now, for any signal \( \tilde{x}[n] \), defined for all \( n \), \( p_N[n] \) is used to choose which \( N \) samples to include in the analysis

\[
x[n] = \tilde{x}[n] p_N[n] = \begin{cases} 
\tilde{x}[n] & -\frac{N}{2} + 1 \leq n \leq \frac{N}{2} \\
0 & \text{otherwise}
\end{cases}.
\]  

(2.8)

The asymmetric window in (2.7) is used to select which \( N \) samples of the received sequences to be used in calculating the CCF. The resulting CCF will consist of \( M \leq 2N - 1 \) non-zero samples which is selected by a symmetric rectangular window denoted \( p'_M[m] \) defined as

\[
p'_M[m] = \begin{cases} 
1 & -N + 1 \leq m \leq N - 1 \\
0 & \text{otherwise}
\end{cases}.
\]  

(2.9)

That is, the subindex denotes the length of the window and the apostrophe shows whether or not the window is symmetric. Note the difference between \( p_A[n] \) and \( p'_A/2[n] \), that is \( p_A[A/2] \neq p'_A/2[A/2] \).
2.4 Channel model

In a practical multi-channel receiver system the individual noises are, by a practical rule-of-thumb, assumed uncorrelated if the distance between the sensors is larger than ten wavelengths. The considered electronic warfare scenario with receiving sensors separated up to some 1000 km fulfills this for all considered wavelengths and thus the noises are assumed uncorrelated. For simplicity, the noises are also assumed to be equal power and additive white Gaussian. In a practical electronic warfare scenario the acquisition interval is short and thus the effects of fading are ignored. Considering audio or radio channels the propagation speed is independent of the frequency, that is non-dispersive propagation is assumed. The considered models can be extended to include the effects of multipath propagation but analysis of such models are beyond the scope of this thesis. To summarize, the channels are assumed to be time-invariant channels with uncorrelated additive white complex-valued Gaussian noises.

2.5 A baseband model

The baseband model is used when baseband signals are acquired, such as low frequency radio, audio and sonar applications. In this model the outputs from two ideal digital receiving sensors are described by

\[ \tilde{r}_1 [n] = s [n] + z_1 [n] \]  \hspace{1cm} (2.10)

and

\[ \tilde{r}_2 [n] = s [n - \Delta] + z_2 [n] \]  \hspace{1cm} (2.11)

where \( \Delta = \Delta_s f_s \) samples defines the TDOA for the signal of interest between the receiving sensors. The fractional delay of the signal in (2.11) is only used for analysis of the received sequences as discussed in Chapter 2.2. The signal \( s [n] \) is assumed to be a wideband wide sense stationary signal which is uncorrelated to the mutually uncorrelated complex-valued noise sequences \( z_1 [n] \) and \( z_2 [n] \). Also, the noises are assumed to have equal power \( \sigma_z^2 \). To model the limited acquisition interval the sensor outputs are windowed using the rectangular window \( p_N [n] \) in (2.7) resulting in sequences that are non-zero for \( N \) samples each. The windowed received sequences are then given by

\[ r_1 [n] = (s [n] + z_1 [n]) p_N [n] \]  \hspace{1cm} (2.12)

and

\[ r_2 [n] = (s [n - \Delta] + z_2 [n]) p_N [n] . \]  \hspace{1cm} (2.13)
Note that in (2.12)-(2.13) both the signal, \( s[n] \), and the noises, \( z_1[n] \) and \( z_1[n] \), are wide sense stationary (WSS) random processes. However, the received sequences, \( r_1[n] \) and \( r_2[n] \), are not wide sense stationary resulting in a cross-correlation function (CCF) that depends not only on the lag \( m \) but also on the time \( n \). Now, the CCF between the two receiver outputs (2.12)-(2.13) is given by

\[
E \{ r_1[n + m] r_2^*[n] \} = E \{ s[n + m] s^*[n - \Delta] \} p_N[n + m] p_N[n]
\]

where the first equality follows from the assumption of mutually uncorrelated noises and that the signal is uncorrelated to the noises as described in Chapter 2.4. The second equality in (2.14) follows from the definition of the auto-correlation function. The function in (2.14) is dependent on time \( n \) due to the rectangular windows \( p_N[n + m] p_N[n] \) from the limited acquisition interval. To eliminate the dependence of time, the CCF is time-averaged following the strategy in [19]. That is, the time dependence is eliminated by averaging the CCF in (2.14) over time

\[
\phi[m] = \frac{1}{N} \sum_{n=-\infty}^{\infty} E \{ r_1[n + m] r_2^*[n] \}
\]

\[
= \phi_s[m + \Delta] \frac{1}{N} \sum_{n=-\infty}^{\infty} p_N[n] p_N[n + m] \quad (2.15)
\]

for all \( m < \infty \). The CCF in (2.15) is only non-zero for combinations of \( n, m \) such that the windowing functions within the summation is non-zero. Let

\[
P[m] = \frac{1}{N} \sum_{n=-\infty}^{\infty} p_N[n] p_N[n + m] \quad (2.16)
\]

then, for \( M = 2N - 1 \) and \( p'_M[m] \) as defined in (2.9)

\[
P[m] = \frac{1}{N} \sum_{n=-N/2+1}^{N/2} p_N[n + m] = \frac{N - |m|}{N} p'_M[m]. \quad (2.17)
\]

The time-averaged CCF between two time discrete wide sense stationary sequences, received using ideal baseband receivers with a limited acquisition interval, is then given by inserting (2.17) into (2.15)

\[
\phi[m] = \frac{N - |m|}{N} \phi_s[m + \Delta] p'_M[m]. \quad (2.18)
\]
The limited acquisition interval imposes a constraint on the TDOA range of $\Delta$ that is covered by a non-zero $\phi[m]$, which follows from the limited length of the received sequences $r_1[n]$ and $r_2[n]$. To obtain a non-zero CCF, the received sequences must contain a common part of the signal which is true only if $|\Delta| \leq N - 1$. Note that the amplitude of the CCF in (2.18) is reduced due to the factor $(N - |m|)/N$ which for large $|m|$ is close to zero. When estimating the CCF, $|m|$ needs to be significantly smaller than $N$ for the CCF in (2.18) to give reliable results.

To obtain the frequency-domain TDOA model, the cross spectral density is calculated using the discrete Fourier transform (DFT) of the CCF. The DFT of length $M = 2N - 1$ samples is denoted $\mathcal{F}_M \{ \cdot \}$ and is given by the Fourier transform evaluated on a frequency grid $\omega_k = 2\pi k/M$ for the discrete frequency bins $k = 1 - N, ..., N - 1$ [17]. The frequency discrete cross spectral density $\Phi[k]$ of the time-discrete CCF is then given by

$$
\Phi[k] = \mathcal{F}_M \{ \phi[m] \} = \sum_{m=1-N}^{N-1} \phi[m] e^{-j2\pi km/M} \quad k = 1 - N, ..., N - 1.
$$

(2.19)

The time and frequency discrete model is now obtained by inserting (2.18) into (2.19)

$$
\Phi[k] = \sum_{m=-N}^{N-1} \frac{N - |m|}{N} \phi_s[m + \Delta] e^{-j2\pi km/M} \quad k = 1 - N, ..., N - 1.
$$

(2.20)

The cross spectral density in (2.20) is not easily evaluated for a general source signal with an auto-correlation function given by $\phi_s[m]$. To get some insight into the behavior of the cross spectral density, an example signal is considered.

**White signal of interest** This example signal is a complex-valued zero-mean white Gaussian noise sequence with

$$
\phi_s[m] = \sigma_s^2 \delta[m]
$$

(2.21)

where $\delta[m]$ is the discrete Kronecker delta defined as

$$
\delta[m] = \begin{cases} 
1 & m = 0 \\
0 & m \neq 0 
\end{cases}
$$

(2.22)

A military tactical communication system which uses direct-sequence spread-spectrum with long and orthogonal codes have similar characteristics as this example signal.
2.6 Reference imperfections

Now, inserting (2.21) into (2.20) with a TDOA of \( \Delta \) samples gives

\[
\Phi[k] = \frac{N - |\Delta|}{N} \sigma^2 e^{j2\pi k \Delta/M} \quad k = 1 - N, ..., N - 1. \quad (2.23)
\]

The phase of \( \Phi[k] \) is linear in frequency

\[
\Gamma[k] = \text{angle} \Phi[k] = \frac{2\pi k \Delta}{M} \quad k = 1 - N, ..., N - 1 \quad (2.24)
\]

and the TDOA \( \Delta \) can be estimated as previously described by fitting a straight line to the phase curve. In order to analyze and compare different models and estimators in the following, the example white noise signal is used. From a strict mathematical point, \( \Delta \) must be an integer to give the result in (2.23). However, as discussed in Chapter 2.2, the non-integer \( \Delta \) is not a problem in this analysis since the considered signals are digitized versions of continuous signals where no integer constraints are made on the TDOA.

### 2.6 Reference imperfections

In order to measure a TDOA between two digital radio receivers both receiver systems must have the same time and frequency references to avoid degradation due to reference imperfections. However, the receivers are separated with several kilometers and can not share a common reference. In practice, this is solved by using a reference module at each of the receiver sites. This reference module, as illustrated in Figure 2.1, is assumed to be controlled by an external reference source. For simplicity, assume that this external source is the NAVSTAR-GPS system. The outputs of the considered reference modules, described in Table 2.1, are the time reference signal which gives the absolute starting time of the acquisition, the receiver oscillator reference (frequency and phase) used in the superheterodyne receiver to mix the signal to near baseband, and the sampling frequency.

<table>
<thead>
<tr>
<th>Output signal</th>
<th>Signal type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time reference</td>
<td>( t_{\text{trig}} )</td>
</tr>
<tr>
<td>Frequency reference</td>
<td>( f_{\text{ref}}, \varphi_{\text{ref}} )</td>
</tr>
<tr>
<td>Sampling frequency</td>
<td>( f_s )</td>
</tr>
</tbody>
</table>

Table 2.1: The outputs from the reference module.
Figure 2.1: A typical reference module with outputs used in the spatially separated digital receivers to obtain a near time and frequency synchronous TDOA direction-finding system.

High accuracy is achieved using the NAVSTAR-GPS system but, still, small differences between the outputs of the different reference modules occurs which degrades the performance of the system as will be described in Sections 3.4 and 3.5, respectively. From a typical high-end NAVSTAR-GPS receiver [8] the timing error is < 50 ns and the frequency error is < 10^{-10} relative the 10 MHz output signal.

If the GPS receiver fails, the system performance will quickly degrade since both time and frequency reference signals diverges from the external reference. Of course, another way of keeping the time and frequency references accurate is to use an atomic clock which does not rely on an external source. The main disadvantage of the needed high-end atomic clock is its price which is approximately 500 times higher than that for a GPS receiver. Below the errors between the reference modules are discussed in some detail.

2.6.1 Time reference error

In a TDOA-based electronic warfare direction-finding system the TDOA between the outputs of two spatially separated receivers give a direction of arrival for the received signal as shown in previous chapters. A difference between the time reference signals between the two receiver systems produces a biased estimate. This bias is directly linked to the time-difference, or error, between the time reference outputs of the two reference modules. If the timing error is larger than the acquisition interval – the CCF is zero since no common part of the signal exist in the two received sequences. The effects on the system models suffering from a timing error are modelled in Chapter 2.7.3.
2.6.2 Receiver oscillator reference errors

In order to digitize, or sample, the signal it needs to be transposed, or mixed, to near baseband using a superheterodyne receiver where the signal is mixed with a high frequency carrier. This reference frequency signal is controlled by the reference module and differences, or errors, between the outputs of the reference modules will affect the TDOA estimation process. The oscillator phase error is modelled in Chapter 2.7.4 and the frequency error in Chapter 2.7.5.

2.6.3 Sampling frequency reference error

An error between the sampling frequencies in the different receiver systems severely degrades the performance due to decorrelation. However, this error is small compared to the receiver oscillator frequency which is created from the reference frequency output. As an example, the error from the reference module is practically the same for both the sampling frequency output and the frequency reference output. However, the sampling frequency is in most cases small compared to the mixing frequency which is in parity with the carrier frequency of the signal. Consequently the sampling frequency error is smaller than the mixer frequency error. Under this assumption, the sampling frequency error is not considered in this thesis.

2.7 Receiver system models

In a practical electronic warfare system, a number of factors degrades the performance of the system. Some of these factors are wave distortion due to non-ideal antenna placement, analog receiver filter differences resulting in a dispersive receiver system, and the analog to digital converter errors including time and amplitude jitter. However, in this thesis we concentrate on the problems which arises from imperfections in the receiver systems, such as time and frequency reference errors. In the following, models are derived considering the reference imperfections followed by an analysis of its effects on the TDOA estimation process.

In modern electronic warfare direction-finding systems digital wideband receivers are used to acquire the signal as illustrated in Figure 2.2. In this thesis positioning in general and direction-finding in particular is considered and by using two receivers direction-finding is possible, while three or more receivers are needed to calculate a position of the signal. Each of the digital receivers are assumed to consist of an ideal IQ-modulator
followed by an analog superheterodyne receiver and an ideal analog to digital converter as seen in Figure 2.3. The analog superheterodyne receivers are modelled by an input bandpass filter of bandwidth $2W$ Hz followed by the mixer. The mixer frequency is generated using the reference frequency signal which is one of the outputs from the reference module described in Chapter 2.6. Differences between the reference frequency signals used in the different receivers results in a performance degradation which is also described in Sections 2.7.5 and 3.5. Low-pass filtering with bandwidth $\pm W$ Hz of the mixer output gives the complex-valued near baseband component as illustrated in Figure 2.4.

In this thesis, the analysis is concentrated on one pair of receivers. When using more than two receivers the received data sequences are still analyzed in pairs. The signal of interest $s(t)$, which is transmitted at carrier frequency $f_0$ Hz and phase $\phi_0$, is received at the input of the two receivers. After IQ-modulation the received signals

$$r_1(t) = s(t) e^{j2\pi f_0 t + j\phi_0} + z_1(t)$$  (2.25)
2.7. Receiver system models

Figure 2.3: The digital receivers are modelled by an ideal IQ-modulator followed by the analog superheterodyne receiver and the analog to digital converter. The digital output describes a complex-valued near baseband signal.

Figure 2.4: The analog superheterodyne receiver is modelled as a filter-mixer-filter receiver where the mixer frequency is created using the reference frequency from the reference module.
and
\[ r_2 (t) = s (t - \Delta_t) e^{j2\pi f_0 (t - \Delta_t) + j\varphi_0} + z_2 (t) \]  
(2.26)

where \( z_1 (t) \) and \( z_2 (t) \) denotes the additive noises, are fed to the superheterodyne receiver. The goal is to mix these signals to near baseband which is achieved by setting the receiver mixing frequencies \( f_1 \) and \( f_2 \) with phases \( \varphi_1 \) and \( \varphi_2 \), respectively, to near \( f_0 \). This results in
\[
\begin{align*}
    r_1 (t) &= \left\{ s (t) e^{j2\pi f_0 t + j\varphi_0} + z_1 (t) \right\} e^{-j2\pi f_1 t - j\varphi_1} \\
    &= s (t) e^{j2\pi (f_0 - f_1) t + j(\varphi_0 - \varphi_1)} + z_1 (t) e^{-j2\pi f_1 t - j\varphi_1} \\
    & \quad \text{(2.27)}
\end{align*}
\]

and
\[
\begin{align*}
    r_2 (t) &= \left\{ s (t - \Delta_t) e^{j2\pi f_0 (t - \Delta_t) + j\varphi_0} + z_2 (t) \right\} e^{-j2\pi f_2 t - j\varphi_2} \\
    &= s (t - \Delta_t) e^{-j2\pi f_0 \Delta_t + j2\pi (f_0 - f_2) t + j(\varphi_0 - \varphi_2)} + z_2 (t) e^{-j2\pi f_2 t - j\varphi_2} \\
    & \quad \text{(2.28)}
\end{align*}
\]

Following the mixer is the analog-to-digital converter where the received signals are digitized. In the following the discrete frequencies are given relative the sampling frequency \( f_s \). That is, the transmitter frequency is now given by \( f_0 = f_0 f_s^{-1} \) and the receiver mixing frequencies are given by \( f_1 = f_1 f_s^{-1} \) and \( f_2 = f_2 f_s^{-1} \), respectively. The outputs of the digital receivers in Figure 2.3 are then for input signals (2.27)-(2.28) given by
\[
\begin{align*}
    \tilde{r}_1 [n] &= \left\{ s [n] e^{j2\pi \mu_0 n + j\varphi_0} \right\} e^{-j2\pi \mu_1 n - j\varphi_1} + z_1 [n] \\
    &= s [n] e^{j2\pi (\mu_0 - \mu_1) n + j(\varphi_0 - \varphi_1)} + z_1 [n] \\
    & \quad \text{(2.29)}
\end{align*}
\]

and
\[
\begin{align*}
    \tilde{r}_2 [n] &= \left\{ s [n - \Delta] e^{j2\pi \mu_0 (n - \Delta) + j\varphi_0} \right\} e^{-j2\pi \mu_2 n - j\varphi_2} + z_2 [n] \\
    &= s [n - \Delta] e^{-j2\pi \mu_0 \Delta + j2\pi (\mu_0 - \mu_2) n + j(\varphi_0 - \varphi_2)} + z_2 [n] \\
    & \quad \text{(2.30)}
\end{align*}
\]

where the digitized representation of the noises include the complex rotation due to the mixing. The complex rotation will not affect the statistical properties of the noises and is ignored in the following. In a practical receiver system only a limited number of samples are used from the received sequences. That is, the rectangular window \( p_X [n] \) described in Chapter 2.3 is applied to (2.29) and (2.30) to limit the analysis to \( N \) samples.
2.7. Receiver system models

2.7.1 Ideal receiver system model

In some scenarios the receivers are located close to each other and a common reference module can be used. In such a case, the receivers can be connected by a cable and perfect synchronization is achieved. That is, there are no differences between the reference signals leading to \( \mu_1 = \mu_2 \) and \( \varphi_1 = \varphi_2 \). Moreover, in the ideal receiver system the transmitter frequency and phase are assumed known and the receiver mixing frequency is set to the transmitter frequency \( \mu_0 = \mu_1 \) and phase-locked to the transmitter \( \varphi_0 = \varphi_1 \). Now, (2.29)-(2.30) are evaluated to

\[
r_1[n] = (s[n] + z_1[n]) p_N[n] \tag{2.31}
\]

and

\[
r_2[n] = (s[n - \Delta] e^{-j2\pi\mu_0\Delta} + z_2[n]) p_N[n] \tag{2.32}
\]

which is recognized as the same model as in (2.12)-(2.13) but with a complex rotation \( e^{-j2\pi\mu_0\Delta} \) due to the receiver mixing residual in receiver channel 2. The CCF between the received sequences (2.31)-(2.32) then follows from calculations similar to (2.14)-(2.18).

\[
\phi[m] = \frac{N - |m|}{N} \phi_s [m + \Delta] e^{-j2\pi\mu_0\Delta} p'_M [m]. \tag{2.33}
\]

The cross spectral density is given by the time discrete Fourier transform of (2.33). However, the cross spectral density is not easily evaluated for a general source signal with an auto-correlation function given by \( \phi_s [m] \). Evaluating the cross spectral density for the white example signal (2.21), with similarities to wideband military communication sources, gives

\[
\Phi[k] = \frac{N - |\Delta|}{N} \phi_s^2 e^{j2\pi(k/M + \mu_0)\Delta} \quad k = 1 - N, \ldots, N - 1. \tag{2.34}
\]

The corresponding phase of the cross spectral density \( \Gamma[k] = \angle \Phi[k] \) is then given by

\[
\Gamma[k] = 2\pi \left( \frac{k}{M} + \mu_0 \right) \Delta \quad k = 1 - N, \ldots, N - 1. \tag{2.35}
\]

The mixing residual \( 2\pi\Delta\mu_0 \) will only affect the bias of the phase curve but does not affect the performance of the TDOA estimation process since the bias and slope estimation decouples as shown in Chapter 3.3.
Receiver model for multiple sources

In an electronic warfare scenario, the intercept receivers collect all signals within their bandwidth. Clearly, multisource as well as multipath propagation are common in many practical scenarios. In this chapter, a model for multiple sources is introduced. The number of uncorrelated signals is denoted $U$. As in the previous chapter, deriving a model for the time-averaged CCF is the goal.

Only two receivers are used leading to only one mixer in each receiver for all the signals and the mixer oscillator frequencies in the two receivers are again given by $\mu_1$ and $\mu_2$, respectively. In the same way, the mixer oscillator phases are given by $\varphi_1$ and $\varphi_2$, respectively. The carrier frequency and phase of source $u$ are denoted $\mu_u$ and $\varphi_u$, respectively. Now, the received sequences containing $U$ uncorrelated sources are given by

$$r_1[n] = \sum_{u=1}^{U} s_u[n - \Delta^1_u] e^{j2\pi\mu_u(n - \Delta^1_u) + j\varphi_u - j2\pi\mu_1 n - j\varphi_1} + z_1[n]$$

and

$$r_2[n] = \sum_{u=1}^{U} s_u[n - \Delta^2_u] e^{j2\pi\mu_u(n - \Delta^2_u) + j\varphi_u - j2\pi\mu_2 n - j\varphi_2} + z_2[n]$$

where $\Delta^i_u$ denotes the time-delay, not difference, from source $u$ to receiver $i$. In the ideal receiver case the receiver oscillators are assumed phase and frequency synchronized, that is $\mu_1 = \mu_2$ and $\varphi_1 = \varphi_2$. The sources $s_u[n]$ are assumed zero-mean and mutually uncorrelated, that is $E\{s_u[n] s^*_v[n]\} = 0$ for $u \neq v$. The multisource CCF is now given by

$$E\{r_1[n + m] r_2^*[n]\} = \sum_{u=1}^{U} \phi_{s_u}[m + \Delta_u] e^{j2\pi\mu_u(m + \Delta_u) - j2\pi\mu_1 m} p_N[n + m] p_N[n]$$

(2.38)

where $\Delta_u = \Delta^2_u - \Delta^1_u$ describes the TDOA for source $s_u[n]$ between the two receivers. Without loss of generality, the transmitter frequencies $\mu_u$ are assumed to be equal and known, that is $\mu_u = \mu_0$. If there are different transmitter frequencies this can be modelled in the auto-correlation functions $\phi_{s_u}[m]$ of the individual sources. Under the above assumption
the CCF (2.38) becomes

\[ E \{ r_1[n+m] r_2^*[n] \} = \sum_{u=1}^{U} \phi_{s_u} [m + \Delta_u] e^{j2\pi\mu_0 \Delta_u} p_n [n+m] p_n^*[n] \]

which is equal to a sum of single source models (2.14) with a specific complex rotation for each of the uncorrelated sources. Applying time-averaging to remove the dependence of \( n \) gives the time-averaged multi-source CCF

\[ \phi[m] = \frac{N - |m|}{N} \sum_{u=1}^{U} \phi_{s_u} [m + \Delta_u] e^{j2\pi\mu_0 \Delta_u} p_n^*[m] \]. \quad (2.40) \]

Note that for \( U = 1 \) the expression in (2.40) is reduced to the single source result in (2.33). The expression in (2.40) is valid for \( U \) uncorrelated sources which is a sum of single source CCFs. The practical implication of the result in (2.40) is that the multisource CCF has \( U \) peaks describing the TDOAs for the individual sources. However, when the TDOA of the different sources are about the same, the peaks will overlap in the resulting multisource CCF. In this case the multiple sources can not be separated in the spatial domain. However, both time and frequency-domain can be used in combination with the spatial domain to separate the different sources. That is, the source can be separated if they are non-overlapping in either time, frequency or spatial domain.

Calculating the cross spectral density without any spatial filtering of the multisource CCF in (2.40) gives a sum of the individual cross spectra. Considering \( U \) white noise sources (2.21) with different TDOAs yields

\[ \Phi[k] = \sum_{u=1}^{U} \frac{N - |\Delta_u|}{N} \sigma_{s_u}^2 e^{j2\pi(k/M + \mu_0)\Delta_u} \quad k = 1 - N, \ldots, N - 1. \quad (2.41) \]

The phase of (2.41) is given by \( \Gamma[k] = \angle \Phi[k] \) and shows that the phase is given by an amplitude weighted sum of all the sources. However, spatial filtering (time-domain filtering in [9]) can be used to separate the multiple TDOAs.
2.7.3 Receiver model with time reference error

A difference, or error, between the time reference outputs of the reference modules results in a biased estimate of the TDOA. The received sequences in (2.29)-(2.30) are, for a timing error $\Delta_{\text{ref}}$, given by

$$
r_1[n] = \left( s[n] e^{j2\pi(\mu_0 - \mu_1)n + j(\varphi_0 - \varphi_1)} + z_1[n] \right) p_N[n]
$$

(2.42)

and

$$
r_2[n] = \left( s[n - \Delta - \Delta_{\text{ref}}] e^{-j2\pi\mu_0(\Delta + \Delta_{\text{ref}}) + j2\pi(\mu_0 - \mu_2)n + j(\varphi_0 - \varphi_2)} + z_2[n] \right) p_N[n].
$$

(2.43)

For simplicity, let $\mu_0 = \mu_1 = \mu_2$ and $\varphi_0 = \varphi_1 = \varphi_2$, then the received sequences are

$$
r_1[n] = (s[n] + z_1[n]) p_N[n]
$$

(2.44)

and

$$
r_2[n] = s \left( [n - \Delta - \Delta_{\text{ref}}] e^{-j2\pi\mu_0(\Delta + \Delta_{\text{ref}}) + j2\pi(\mu_0 - \mu_1)n + j(\varphi_0 - \varphi_1)} + z_2[n] \right) p_N[n].
$$

(2.45)

In this case, the time-averaged CCF follows directly from the calculations presented in Chapter 2.7.1 with $\Delta$ replaced by $\Delta + \Delta_{\text{ref}}$. The result reads

$$
\phi[m] = \frac{N - |m|}{N} \phi_s [m + \Delta + \Delta_{\text{ref}}] e^{j2\pi\mu_0(\Delta + \Delta_{\text{ref}}) + j2\pi(\mu_0 - \mu_1)n + j(\varphi_0 - \varphi_1)} p_M'[m].
$$

(2.46)

Accordingly, the peak of the averaged CCF is translated from the actual TDOA. Note that the timing error results in an extra amplitude reduction as seen in (2.34) for $\Delta \rightarrow \Delta + \Delta_{\text{ref}}$.

2.7.4 Receiver model with oscillator phase error

The difference in phase between the mixer frequency reference outputs using two reference modules affects the model but not the performance in terms of TDOA estimation as seen in the following. The reference oscillator phase difference denoted $\gamma$ is defined as the difference between the phases of the individual oscillators, $\gamma = \varphi_2 - \varphi_1$ and the received sequences suffering from an oscillator phase error are given by

$$
r_1[n] = \left( s[n] e^{j2\pi(\mu_0 - \mu_1)n + j(\varphi_0 - \varphi_1)} + z_1[n] \right) p_N[n]
$$

(2.47)

and

$$
r_2[n] = \left( s[n - \Delta] e^{-j2\pi\mu_0(\Delta + \Delta_{\text{ref}}) + j2\pi(\mu_0 - \mu_2)n + j(\varphi_0 - \varphi_2)} + z_2[n] \right) p_N[n].
$$

(2.48)
2.7. Receiver system models

To simplify the analysis, let \( \mu_0 = \mu_1 = \mu_2 \). Once again, similar calculations as (2.14)-(2.18) results in

\[
\phi[m] = \frac{N - |m|}{N} \phi_s[m + \Delta] e^{j2\pi \mu_0 \Delta + j\gamma} p_M[m]. \tag{2.49}
\]

The complex rotation caused by the receiver oscillator phase difference \( \gamma \) does not affect the magnitude of \( \phi[m] \). Thus, it will not affect the TDOA estimators based on the magnitude of (2.49).

In frequency domain, this is illustrated by the white example signal in (2.21). The spectral properties are now

\[
\Phi[k] = \frac{N - |\Delta|}{N} \sigma_s^2 e^{j2\pi(k + \mu_0)\Delta + j\gamma} \quad k = 1 - N, \ldots, N - 1 \tag{2.50}
\]

and

\[
\Gamma[k] = \angle \Phi[k] = 2\pi \left( \frac{k}{N} + \mu_0 \right) \Delta + \gamma \quad k = 1 - N, \ldots, N - 1. \tag{2.51}
\]

The phase error \( \gamma \) results in a bias of the phase curve but will not affect the slope of the phase curve. Thus, the receiver oscillator phases do not affect the result in terms of TDOA estimation. A result which follows from this, is that the absolute phases of the transmitter or the receivers will not affect the performance of the considered TDOA estimators.

2.7.5 Receiver model with oscillator frequency error

The receiver oscillator frequency error is due to the frequency difference between the frequency reference outputs of two different reference modules. Accordingly, the receiver frequency imperfections result in a frequency shift which must not be confused with doppler (time scaling) introduced by moving transmitters or receivers [5]. For simplicity, assume the transmitter frequency and phase to be known, \( \mu_0 = \mu_1 \) and \( \varphi_0 = \varphi_1 = \varphi_2 \), respectively. A frequency difference, or error, between the two receiver oscillators is defined as \( \varepsilon = \mu_2 - \mu_1 \) which here is assumed small, that is \( \varepsilon \ll 1/N \). The effects of large \( \varepsilon \) are considered in Chapter 3.6. Now, the received sequences are

\[
r_1[n] = (s[n] + z_1[n]) p_N[n] \tag{2.52}
\]

and

\[
r_2[n] = \left( s[n - \Delta] e^{-j2\pi \mu_0 \Delta + j2\pi (\mu_0 - \mu_2)n} + z_2[n] \right) p_N[n] = (s[n - \Delta] e^{-j2\pi \mu_0 \Delta - j2\pi \varepsilon n} + z_2[n]) p_N[n]. \tag{2.53}
\]
The CCF between (2.52) and (2.53) is

\[ E\{r_1[n + m]r_2^*[n]\} = \phi_s [m + \Delta] e^{j2\pi\mu_0\Delta} \frac{1}{N} \sum_{n=1-N}^{N-1} e^{j2\pi\varepsilon n} p_N[n + m] p_N[n] \]

which is dependent on time \( n \). Time-averaging is performed in accordance with (2.18)

\[ \phi [m] = \phi_s [m + \Delta] e^{j2\pi\mu_0\Delta} \frac{1}{N} \sum_{n=-N/2+1}^{N/2} e^{j2\pi\varepsilon n} p_N [n + m]. \quad (2.55) \]

An explicit formula for the sum in (2.55) is given by Appendix A.1. The resulting CCF is

\[ \phi [m] = \frac{N - |m|}{N} \phi_s [m + \Delta] e^{j2\pi\mu_0\Delta + j\pi\varepsilon(1 - m)} \frac{\text{sinc} (\varepsilon (N - |m|))}{\text{sinc} (\varepsilon)} p_M [m]. \quad (2.56) \]

Again, considering the white example signal in (2.21), the cross spectral density of (2.56) is given by

\[ \Phi [k] = \frac{N - |\Delta|}{N} \sigma_s^2 e^{j2\pi k/M + j2\pi\mu_0\Delta + j\pi\varepsilon(1 + \Delta)} \frac{\text{sinc} (\varepsilon (N - |\Delta|))}{\text{sinc} (\varepsilon)} \]

for \( k = 1 - N, ..., N - 1 \). The phase of (2.57) is then

\[ \Gamma [k] = \frac{2\pi k\Delta}{M} + 2\pi\mu_0\Delta + \pi\varepsilon (1 + \Delta) \quad k = 1 - N, ..., N - 1 \quad (2.58) \]

where it is seen that the receiver oscillator frequency error gives a biased phase curve which not does affect the slope. However, as seen in (2.56) and (2.57) the sinc-factor results in an amplitude reduction given \( \varepsilon, \Delta \) and \( N \). This amplitude reduction leads to a noise-like degradation in the estimation process, as shown in Chapter 3. The robustness against frequency errors is further analyzed in Chapter 3.6 where the effects of large frequency errors also are studied.
2.8 Summary of models

Several models of a two channel TDOA direction-finding system are derived and studied. In Table 2.2 the key properties of the considered models are presented. The baseband model is to be used with non-mixing receivers, such as low frequency radio, audio and sonar applications. The ideal receiver system model is used when the considered mixing receivers are controlled by the same reference module. When the mixing receivers use separate reference modules the effects of errors between the outputs of the reference modules are considered in the receiver system models subject to timing error, oscillator phase error and oscillator frequency error.

<table>
<thead>
<tr>
<th>Model</th>
<th>Variable</th>
<th>$\phi[m]$</th>
<th>$\Gamma[k]$ for example signal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseband</td>
<td>–</td>
<td>(2.18)</td>
<td>$2\pi \phi^{1\over T} \Delta$</td>
</tr>
<tr>
<td>Ideal receivers</td>
<td>$\mu_0$</td>
<td>(2.33)</td>
<td>$2\pi (\phi^{1\over T} + \mu_0) \Delta$</td>
</tr>
<tr>
<td>Timing error</td>
<td>$\Delta_{ref}$</td>
<td>(2.46)</td>
<td>$2\pi (\phi^{1\over T} + \mu_0) (\Delta + \Delta_{ref})$</td>
</tr>
<tr>
<td>Phase error</td>
<td>$\gamma$</td>
<td>(2.49)</td>
<td>$2\pi (\phi^{1\over T} + \mu_0) \Delta + \gamma$</td>
</tr>
<tr>
<td>Frequency error</td>
<td>$\varepsilon$</td>
<td>(2.56)</td>
<td>$2\pi (\phi^{1\over T} + \mu_0) \Delta + \pi \varepsilon (1 + \Delta)$</td>
</tr>
</tbody>
</table>

Table 2.2: Summary of system models
Chapter 3

Time- and frequency-domain TDOA estimation

Correlation based TDOA estimation can be made using time- or frequency-domain estimators. In previous work it is shown that these estimators have similar characteristics and that both the time-domain estimator [14] and the frequency-domain estimator [18] are unbiased and attain the Cramér-Rao lower bound (CRLB) for large enough data records (asymptotically efficient). In an electronic warfare scenario, suppression of unwanted signals is an important feature. Using a frequency-domain estimator yields simple frequency filtering while a time-domain estimator allows simple spatial filtering. Promising results from measurements of actual radio transmitters using the considered estimators are described in [25]. In a practical TDOA-based direction-finding system, a combined time- and frequency-domain estimator can be used. That is, the CCF is estimated and spatial filtering is applied [9]. The filtered CCF is then used to calculate the estimated cross spectral density where frequency filtering is performed. In terms of bias and variance the performance is similar between the time- and frequency-domain estimators. In this thesis, the focus is on frequency-domain estimators due the simple frequency filtering which is needed in electronic warfare scenarios.

3.1 A lower bound on TDOA estimation accuracy

A well known and commonly used lower bound on the variance in estimation problems is the CRLB [13]. The extensive derivation of the CRLB for correlation based TDOA estimation is presented in [14] with examples
of its application given in [18] and [26]. In this thesis the main focus is on time discrete models but for simplicity the following calculation of the CRLB is made for a time continuous model. Basically the CRLB is given by [14]

\[
\text{CRLB}(\Delta t) = \frac{1}{8\pi^2 T^2} \int_{-\infty}^{\infty} \frac{f^2 C(f)}{1 - C(f)} df \right]^{-1} \quad (3.1)
\]

where \( C(f) \) is the squared coherence function defined by the cross spectral density and the power spectral density of the sensor outputs, \( \Phi_1(f) \) and \( \Phi_2(f) \), as

\[
C(f) = \frac{|\Phi(f)|^2}{\Phi_1(f) \Phi_2(f)}. \quad (3.2)
\]

Under an assumption of equal power in the two sensor outputs in bandlimited additive white complex-valued Gaussian noise with variance \( \sigma_z^2 \), the squared coherence function is for any model described in Chapter 2 given by

\[
C(f) = \frac{\Phi_s^2(f)}{[\Phi_s(f) + \sigma_z^2]^2}. \quad (3.3)
\]

The SNR between the signal and the channel noise is defined as

\[
\text{SNR}(f) = \frac{\Phi_s(f)}{\sigma_z^2}. \quad (3.4)
\]

where \( \Phi_s(f) \) denotes the power spectral density of the signal of interest \( s(t) \). For example, a flat spectrum signal with power \( \sigma_s^2 \) results in a constant SNR within the bandwidth of the noise

\[
\text{SNR} = \frac{\sigma_s^2}{\sigma_z^2} \quad (3.5)
\]

and (3.3) is evaluated to

\[
C(f) = \frac{\text{SNR}^2}{[\text{SNR} + 1]^2}. \quad (3.6)
\]

Inserting (3.6) into (3.1) gives the CRLB for a flat spectrum signal in additive white complex-valued Gaussian noise

\[
\text{CRLB}(\Delta t) = \frac{1}{8\pi^2 T^2} \frac{1 + 2\text{SNR}}{\text{SNR}^2} \left[ \int_{-W}^{W} f^2 df \right]^{-1}
\]

\[
= \frac{3}{16\pi^2 TW^3} \frac{1 + 2\text{SNR}}{\text{SNR}^2}. \quad (3.7)
\]
3.1. A lower bound on TDOA estimation accuracy

The CRLB in (3.7) is evaluated over the signal bandwidth - 2W Hz. Note that the CRLB decreases with $W^3$ while it is linear in $T$ and SNR as illustrated in Figure 3.1 for the white example signal. In [11], it is shown that the variance of a traditional phase-measuring direction-finding system decreases with $W$. This implies that a correlation based direction-finding system potentially outperforms a traditional phase-measuring system for wideband signals.

For sampled data, the CRLB for $\Delta = \Delta t f_s$ is sought. Straightforward calculations give

$$CRLB (\Delta) = f_s^2 CRLB (\Delta t).$$  \hfill (3.8)

Considering Nyquist sampling with $f_s = 2W$ yields by inserting $f_s = 2W$ into (3.7)

$$CRLB (\Delta) = \frac{3}{2 \pi^2 T f_s} \frac{2SNR + 1}{SNR^2}. \hfill (3.9)$$

Finally, observing that the number of samples $N$ equals $N = T f_s$ results

Figure 3.1: The CRLB (3.7) for the example signal in AWGN is plotted for different acquisition times $T$ and bandwidths $W$. 
in
\[ \text{CRLB} (\Delta) = \frac{3}{2\pi^2 N} \frac{2\text{SNR} + 1}{\text{SNR}^2}. \] (3.10)

That is, the CRLB for a flat spectrum signal, using Nyquist sampling, is for \( N \) samples given by (3.10). The CRLB can be calculated for any signal with known spectral characteristics (3.2). In a practical system, targeting military communication systems, the acquisition interval and signal bandwidth are in the range \( T > 1 \text{ ms} \) and \( W > 100 \text{ kHz} \). The range of the CRLB is then larger than \( 10^{10} \) considering typical military communication systems as illustrated in Figure 3.1.

### 3.2 Time-domain TDOA estimator

In 1976, the generalized method for estimation of time delay using the generalized CCF was introduced [14]. This method is considered a standard reference by most researchers in this field. The basic idea of the generalized CCF is to compare, or correlate, the outputs of two sensors and determine the time delay between the two channels. Alternatives to the generalized CCF method are the average-square-difference function and the average-magnitude-difference function [10],[16] which both are less complex to calculate than the generalized CCF. For medium and high SNR the two alternative methods show results in parity with the considered methods based on the generalized CCF. However, for low SNR the alternative methods fail while the generalized CCF produces usable results. Accordingly, in the considered electronic warfare scenario, under the low SNR assumption, the generalized CCF method is considered.

The generalized TDOA method presented in [14] is reviewed in Chapter 1.2. However, in this thesis, the effects of a limited acquisition interval for time-discrete data are considered. The method in [14] is adjusted for these effects in the following. Also, with the electronic warfare scenario no signal characteristics are known and the estimator needs to function without any a priori information of the signal. The considered estimators of \( \Delta \) rely on an estimate of the cross-correlation function, that is
\[
\hat{\phi} [m] = \frac{1}{N} \sum_{n=1-N}^{N-1} r_1 [n + m] r_2^* [n]. \] (3.11)

Below it is shown that (3.11) is an unbiased estimate of the time-averaged CCF in (2.18) for the baseband model (2.12)-(2.13). The ex-
3.2. Time-domain TDOA estimator

<table>
<thead>
<tr>
<th>Step</th>
<th>Procedure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Acquire the outputs from two spatially separated sensors, ( r_1[n] ) and ( r_2[n] ) of length ( N ) samples each.</td>
</tr>
<tr>
<td>2.</td>
<td>Calculate ( \hat{\phi}[m] ) given by (3.11).</td>
</tr>
<tr>
<td>3.</td>
<td>Solve (3.14) and denote the result ( \hat{\Delta}_m ).</td>
</tr>
<tr>
<td>4.</td>
<td>Fit a second degree polynomial ( \Pi(\Delta) ) to ( \hat{\Delta}<em>m ) and its two nearest neighbors, ( \hat{\Delta}</em>{m-1} ) and ( \hat{\Delta}_{m+1} ).</td>
</tr>
<tr>
<td>5.</td>
<td>Calculate the refined estimate using ( \hat{\Delta} = \arg \max \Pi(\hat{\Delta}) ) or ( \frac{\partial \Pi(\hat{\Delta})}{\partial \hat{\Delta}} = 0 ).</td>
</tr>
</tbody>
</table>

Table 3.1: An outline for a time domain TDOA estimator.

The expected value of (3.11) is calculated by inserting (2.12)-(2.13) into (3.11)

\[
E \{ \hat{\phi}[m] \} = \frac{1}{N} \sum_{n=1-N}^{N-1} E \{ s[n+m] s^*[n-\Delta] \} p_N[n+m] p_N[n] \\
= \phi_s[m+\Delta] \frac{1}{N} \sum_{n=1-N}^{N-1} p_N[n+m] p_N[n] 
\]

(3.12)

where \( \phi_s[m] = E \{ s[n+m] s^*[n] \} \). The definition and closed form expression of \( P[m] \) in (2.16)-(2.17) is used to evaluate (3.12) and it is seen that this estimator of the CCF is an unbiased estimator of the time-averaged CCF in (2.18)

\[
E \{ \hat{\phi}[m] \} = \frac{N-|m|}{N} \phi_s[m+\Delta] p_M[m] \\
= \phi[m]. 
\]

(3.13)

That is, the considered time-domain CCF estimator in (3.11) is unbiased.

The time-domain TDOA estimator is given by the maximizing argument of the estimated CCF

\[
\hat{\Delta} = - \arg \max_m |\hat{\phi}[m]|. 
\]

(3.14)

An outline for estimating the TDOA using a time-domain estimator is given in Table 3.1. The estimator in (3.14) provides a rough estimate of \( \Delta \). Note that \( \hat{\phi}[m] \) is an unbiased estimate of \( \phi[m] \), while \( \hat{\Delta} \) is in its simplest form biased due to the time-discrete CCF. The performance in terms of mean square error (MSE) of the time-domain TDOA estimator.
Figure 3.2: The MSE for the example signal is plotted using the time-domain TDOA estimator for $\Delta = 0$ and $\Delta = 500$ samples using $N = 1024$ samples and 500 Monte-Carlo simulations. The CRLB is given for comparison.

in (3.14) is for the white example signal presented in Figure 3.2 as a function of the SNR. For low SNR, the estimator performance deviates from the CRLB, and a sharp delay-dependent threshold effect is observed. For SNRs below the threshold, typically the estimator identifies a peak in the CCF originating from the noise as the maximum peak. When this happens the variance of the time-domain estimator will be much larger than the CRLB due to the large errors [24] which can be seen in Figure 3.2 for low SNRs where the MSE differs from the CRLB. In the considered electronic warfare scenario the acquisition interval is large but the SNR is low causing large estimation errors. The method of block-averaging described in Chapter 3.3.3 can be used to suppress the effects of the noise in the estimated CCF, that is lower the variance of $\hat{\phi}[m]$ to reduce the occurrences of large errors.

Note that in (3.13) the magnitude of the expected CCF is a function of $\Delta$ and an increased delay (positive or negative) reduces the magnitude
3.2. Time-domain TDOA estimator

Due to the factor $(N - |m|)/N$. Thus, the performance of (3.14) is a function of $\Delta$. The CCFs for two realizations of the white example signal with different $\Delta = \{0, 500\}$ is shown in Figure 3.3 from which it is evident that the amplitude reduction for $\Delta = 500$ with $N = 1024$ is drastic and affects the performance of the estimator. According to Figure 3.2, the estimator is statistically efficient for $\Delta = 0$ while an increased $\Delta$ results in an increased estimation error using the white example signal (2.21). In this example the amplitude loss is approximately $|\Delta|/N = 500/1024$, or $\approx 3$ dB. However, for $N \to \infty$ the estimator in (3.14) is efficient for all limited $\Delta$ since $(N - |m|)/N \to 1$. 

Figure 3.3: The CCF is estimated using the example signal with $\Delta = \{0, 500\}$ and $N = 1024$ samples. Note the reduction in amplitude due to the factor $(N - |\Delta|)/N$ (dotted line).
3.3 Frequency-domain TDOA estimator

The frequency-domain TDOA estimator is based on the estimated cross spectral density \( \hat{\Phi} [k] \) which is calculated using the estimated CCF

\[
\hat{\Phi} [k] = \mathcal{F}_M \{ \hat{\phi} [m] \}
\]

where \( \mathcal{F}_M \{ \cdot \} \) denotes the discrete Fourier transform (DFT) for \( k = 1 - N, \ldots, N - 1 \). For the models in Chapter 2 the phase of the cross spectral density is linear. This can be used in a frequency-domain estimator which estimates the linear slope using a linear least squares estimator (LLSE). Generally the estimated phase curve is modelled as

\[
\hat{\Gamma} [k] = \Gamma [k] + v [k] \quad k = 1 - N, \ldots, N - 1
\]

where \( \Gamma [k] \) is the modelled phase and \( v [k] \) is the disturbance caused by noise, receiver imperfections and the limited acquisition interval. The LLSE is found by minimizing the least squares error cost function \( J (\Delta) \) [4], that is minimizing the quadratic difference between the measured and modelled phase curves. Here, let \( \hat{\gamma} \) denote the bias of the phase from the reference imperfections, such as the time, phase and frequency errors described in Table 2.2. The expected phase curve is then given by

\[
\Gamma [k] = \frac{2\pi k \Delta}{M} + \hat{\gamma} \quad k = 1 - N, \ldots, N - 1
\]

Now, the least squares cost function is given by

\[
J (\Delta) = \sum_{k=1-N}^{N-1} \left( \hat{\Gamma} [k] - \Gamma [k] \right)^2
\]

\[
= \sum_{k=1-N}^{N-1} \left( \hat{\Gamma} [k] - \frac{2\pi k \Delta}{M} - \hat{\gamma} \right)^2.
\]

Note that this cost function is valid for all of the models discussed in this thesis. The minimum of the cost function in (3.18) is found by setting its first derivative to zero. That is,

\[
\frac{\partial J (\hat{\Delta})}{\partial \Delta} = \sum_{k=1-N}^{N-1} -k \hat{\Gamma} [k] + \frac{2\pi k^2 \hat{\Delta}}{M} + k \hat{\gamma} = 0
\]
### 3.3. Frequency-domain TDOA estimator

which solved for $\hat{\Delta}$ gives

$$\hat{\Delta} = \frac{M}{2\pi} \sum_{k=1}^{N-1} \frac{\hat{\Gamma}[k] - \gamma \sum_{k=1}^{N-1} k}{\sum_{k=1}^{N-1} k^2}. \tag{3.20}$$

This estimator is independent of any phase bias since

$$\sum_{k=1}^{N-1} k \equiv 0 \tag{3.21}$$

which gives the frequency-domain TDOA estimator using (4.22)

$$\hat{\Delta} = \frac{3}{2\pi N (N-1)} \sum_{k=1}^{N-1} \hat{\Gamma}[k]. \tag{3.22}$$

A simple outline for estimating the TDOA using this frequency-domain estimator is given in Table 3.2 and is analyzed in the following chapters. In particular, the effects of a limited acquisition interval and the effects of a receiver error are studied.

<table>
<thead>
<tr>
<th>Step</th>
<th>Procedure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Acquire the outputs from two spatially separated sensors, $r_1[n]$ and $r_2[n]$ of length $N$ samples each.</td>
</tr>
<tr>
<td>2.</td>
<td>Calculate $\hat{\phi}[m]$ given by (3.11).</td>
</tr>
<tr>
<td>3.</td>
<td>Calculate $\hat{\Phi}[k]$, which is the DFT of $\hat{\phi}[m]$.</td>
</tr>
<tr>
<td>4.</td>
<td>Calculate the phase curve $\hat{\Gamma}[k] = \angle \hat{\Phi}[k]$</td>
</tr>
<tr>
<td>5.</td>
<td>Estimate $\Delta$ using (3.22).</td>
</tr>
</tbody>
</table>

Table 3.2: An outline for a frequency domain TDOA estimator.

### 3.3.1 Practical aspects

In a practical TDOA direction-finding system the receivers are normally spatially separated several kilometers and the received sequences need to be transmitted to a remote processing station where the digital signal processing takes place. By using the frequency-domain TDOA estimator in Table 3.2, simple frequency filtering is possible and the amount of data to be transmitted is reduced, as shown in the following. The data reduction follows from the method of calculating the estimated cross spectral
density from the DFT of the two received sequences, that is the cross spectral density is estimated using the $M$ sample DFT of the two received $N$ sample sequences

$$
\hat{\Phi}[k] = \frac{1}{N} \mathcal{F}_M \{ r_1[n] \} \mathcal{F}_M \{ r_2^*[n] \}.
$$

(3.23)

The result in (3.23) is well known in spectral estimation and is based on the Wiener-Kinchin theorem [12] stating that the Periodogram can be estimated from either the signal itself or its auto-correlation function. The generalization to the cross spectral density is straightforward, but included for reference. Normally the estimated CSD is calculated for $k = 1 - N \ldots N - 1$ as

$$
\hat{\Phi}[k] = \mathcal{F}_M \left\{ \hat{\phi}[m] \right\} = \frac{1}{N} \sum_{m=1-N}^{N-1} \sum_{n=1-N}^{N} r_1[n + m] r_2^*[n] e^{-j2\pi km/M}.
$$

(3.24)

However, substituting $n + m$ with $q$ in (3.24) gives

$$
\hat{\Phi}[k] = \frac{1}{N} \sum_{n=1-N}^{N-1+n} \sum_{q=1-N+n}^{N-1+n} r_1[q] r_2^*[n] e^{-j2\pi k(q-n)/M}.
$$

(3.25)

The received sequences, $r_1[n]$ and $r_2[n]$, are zero outside the range $-N/2 + 1 \leq n \leq N/2$ due to the limited acquisition interval. Accordingly, the summation

$$
\sum_{q=1-N+n}^{N-1+n} r_1[q] e^{-j2\pi kq/M}
$$

(3.26)

is non-zero only for $-N/2 + 1 \leq q \leq N/2$. That is, the limits of the summation is changed since $1 - N + n \leq -N/2 + 1$ and $N - 1 + n \geq N/2$ must be fulfilled to achieve a non-zero result. The sum in (3.26) is then written as

$$
\sum_{q=1-N}^{N-1} r_1[q] e^{-j2\pi kq/M}.
$$

(3.27)

Now, the CSD in (3.25) is simplified to

$$
\hat{\Phi}[k] = \frac{1}{N} \sum_{n=1-N}^{N-1} \sum_{q=1-N}^{N-1} r_1[q] r_2^*[n] e^{-j2\pi k(q-n)/M}
$$

$$
= \frac{1}{N} \mathcal{F}_M \{ r_1[n] \} \mathcal{F}_M \{ r_2^*[n] \}.
$$

(3.28)
3.3. Frequency-domain TDOA estimator

The estimator in (3.22) uses $\hat{\Gamma}[k] = \angle \hat{\Phi}[k]$ which now is given by

$$\hat{\Gamma}[k] = \angle F_M \{r_1[n]\} - \angle F_M \{r_2[n]\}. \quad (3.29)$$

That is, after applying the DFT locally at the individual receivers to the received sequences, only the phase within the signal bandwidth is needed to calculate the TDOA. Accordingly, the amplitude information is not needed and is not transmitted thus reducing the total amount of transmitted data by 50%.

For low SNR, the estimated cross spectral density suffers from large estimation errors caused by the angle-operator ($\angle$) whose output is limited to $[-\pi, \pi]$. An unwrap-operator in combination with the angle-operator is used to produce an output in the range of $[-\infty, \infty]$ where the unwrap operator unwraps the radian phases changing absolute jumps greater than $\pi$ to their $2\pi$-complement. In Figure 3.4, it is seen that for high SNR the unwrap-operator is successful in producing a correct result, that is
Step | Procedure
--- | ---
1. | Follow steps 1-3 in the procedure outlined in Table 3.1
2. | Form the shifted CCF $\hat{\phi}[k] \rightarrow \hat{\phi}[k + \hat{\Delta}_m]$ to move the peak to lag zero.
3. | Follow steps 3-5 in the procedure outlined in Table 3.2

Table 3.3: An outline for a frequency domain TDOA estimator with reduced unwrapping problems.

a straight line within the signal bandwidth. For low SNR the unwrap operator fails to resolve the limited output of the angle-operator. To reduce the occurrences of unwrapping failure the variance of the estimated slope needs to be reduced, which in Chapter 3.3.3 is achieved using block-averaging. In practice, an additional method can be used to reduce, not eliminate, the unwrapping problem and is briefly presented in Table 3.3. That is, by moving the peak of the CCF to lag zero, step 2 in Table 3.3, the slope of the cross spectral density is limited so that the occurrences of unwrapping failures are reduced. However, large errors still occur for low SNR.

3.3.2 Performance analysis

The performance of the considered frequency-domain TDOA estimator presented in Table 3.2 is analyzed below. It is shown to be unbiased and with variance close to the CRLB for high SNR.

The expected value of the considered frequency-domain estimator follows from (3.22)

$$
E\left\{\hat{\Delta}\right\} = \frac{3}{2\pi N (N - 1)} \sum_{k=1}^{N-1} kE\left\{\hat{\Gamma}[k]\right\}. \tag{3.30}
$$

Using (3.16) the expected value of the phase curve is

$$
E\left\{\hat{\Gamma}[k]\right\} = \angle\Phi[k] \quad k = 1 - N, ..., N - 1 \tag{3.31}
$$

for any of the models given in Table 2.2. Note that the phase curves of the models presented in Table 2.2 are all on the form

$$
\frac{2\pi k\Delta}{M} + \hat{\gamma} \tag{3.32}
$$
where $\hat{\gamma}$ is a bias that includes the effects of the imperfections as discussed around (3.16). Now, using (4.22) and (3.21) the expression in (3.30) is

$$E\left\{ \hat{\Delta} \right\} = \frac{3}{2\pi N (N - 1)} \sum_{k=1}^{N-1} k \left( \frac{2\pi k \Delta}{M} + \hat{\gamma} \right)$$

$$= \frac{\Delta}{N (N - 1) M} \sum_{k=1}^{N-1} k^2 + \hat{\gamma} \sum_{k=1}^{N-1} k.$$  \hspace{1cm} (3.33)

The evaluation of the first sum is found in Appendix A.3 and the last sum is strictly zero leading to

$$E\left\{ \hat{\Delta} \right\} = \Delta.$$  \hspace{1cm} (3.34)

That is, the frequency-domain TDOA estimator is unbiased for all the models presented in this thesis.

In previous work the variance of the frequency-domain TDOA estimator is shown to (asymptotically) attain the CRLB assuming small estimation errors [18]. The following analysis follows [18] but is adjusted for a digitized complex-valued model. Now, the variance of the estimator in (3.22) is calculated for a complex-valued system model

$$\text{var} \left( \hat{\Delta} \right) = \frac{9}{4\pi^2 N^2 (N - 1)^2} \text{var} \left( \sum_{k=1}^{N-1} k \hat{\Gamma} [k] \right).$$  \hspace{1cm} (3.35)

In [4] it is shown that for Gaussian noise and large enough time-bandwidth products ($WT > 8$), the bins of the estimated phase curve $\hat{\Gamma} [k]$ are uncorrelated. Here, the noises are assumed Gaussian and the time-bandwidth condition is easily met in practice when considering wideband signals. Accordingly, the bins of the phase curve are assumed uncorrelated which leads to

$$\text{var} \left( \sum_{k=1}^{N-1} k \hat{\Gamma} [k] \right) = \sum_{k=1}^{N-1} k^2 \text{var} \left( \hat{\Gamma} [k] \right).$$  \hspace{1cm} (3.36)

Now, the variance of $\hat{\Delta}$ is given by

$$\text{var} \left( \hat{\Delta} \right) = \frac{9}{4\pi^2 N^2 (N - 1)^2} \sum_{k=1}^{N-1} k^2 \text{var} \left( \hat{\Gamma} [k] \right).$$  \hspace{1cm} (3.37)

In (3.37) the variance is approximately given by [2]

$$\text{var} \left( \hat{\Gamma} [k] \right) \approx \frac{1 - C_{12} [k]}{C_{12} [k]^{1/2}}.$$  \hspace{1cm} (3.38)
where $C_{12}[k]$ is the (magnitude squared) coherence function given by the cross spectral density and the power spectral densities of the received sequences

$$C_{12}[k] = \frac{|\Phi_1[k]|^2}{\Phi_1[k] \Phi_2[k]}.$$  \hfill (3.39)

In (3.39), $\Phi_1[k]$ and $\Phi_2[k]$ denotes the power spectral densities of the received sequences $r_1[n]$ and $r_2[n]$, respectively. For the white example signal \((2.21)\) used with any of the models derived in Chapter 2, the coherence function in (3.39) is given by

$$C_{12}[k] = \left( \frac{N - |\Delta|}{N} \right)^2 \frac{\sigma_s^4}{(\sigma_s^2 + \sigma_z^2)^2}. \hfill (3.40)$$

where $\sigma_z^2$ denotes the power of the noises. The SNR is then defined as

$$\text{SNR} = \frac{\phi_s[0]}{\sigma_z^2} = \frac{\sigma_s^2}{\sigma_z^2} \hfill (3.41)$$

which inserted into (3.40) gives the coherence function for a flat spectrum signal with equal SNR in the two receiver channels

$$C_{12}[k] = \left( \frac{N - |\Delta|}{N} \right)^2 \frac{\text{SNR}^2}{(\text{SNR} + 1)^2}. \hfill (3.42)$$

The variance of the phase curve in (3.38) is then given by

$$\text{var} \left( \hat{\Gamma}[k] \right) \approx \left( \frac{N - |\Delta|}{N} \right)^{-2} \left[ \frac{2\text{SNR} + 1}{\text{SNR}^2} + 1 - \left( \frac{N - |\Delta|}{N} \right)^2 \right]. \hfill (3.43)$$

which is independent of frequency $k$ since both the signal and noises are flat within the receiver bandwidth. Now, the variance of the considered TDOA estimator is found by inserting (3.43) into (3.37)

$$\text{var} \left( \hat{\Delta} \right) \approx \frac{3}{4\pi^2 N(N-1)} \left( \frac{N - |\Delta|}{N} \right)^{-2} \left[ \frac{2\text{SNR} + 1}{\text{SNR}^2} + 1 - \left( \frac{N - |\Delta|}{N} \right)^2 \right]. \hfill (3.44)$$

The approximation in (3.38), and consequently (3.44), is not valid for large estimation errors (low SNR or small $N$ in combination with large $\Delta$) when the phase curve suffers from unwrapping problems as discussed in Chapter 3.3.1. Note that the amplitude reduction from the limited acquisition interval increases the variance. However, for large acquisition intervals and small TDOAs, (3.44) is close to the CRLB in (3.10).
3.3. Frequency-domain TDOA estimator

3.3.3 Variance reduction using block-averaging

In order to reduce the variance of the estimated cross spectral density, block-averaging is used. Block-averaging is used in a variety of applications, for example in spectral estimation both the averaged periodogram and the Welch periodogram use block-averaging [12]. The block-averaging is performed by dividing the acquired sequences of length \( N \) samples into \( B \) blocks of length \( L \) samples. For simplicity the blocks are assumed non-overlapping but in a practical scenario an increase in performance can be achieved using overlapping blocks. An estimated block-cross spectral density \( \hat{\Phi}_b[k] \) is calculated for the individual blocks, with individual length \( A = 2L - 1 \) samples. The \( B \) blocks are averaged to form an averaged estimate of the cross spectral density

\[
\hat{\Phi}_{avg}[k] = \frac{1}{B} \sum_{b=1}^{B} \hat{\Phi}_b[k] \quad k = 1 - L, ..., L - 1 \quad (3.45)
\]

where \( \hat{\Phi}_b[k] \) is given by (3.15), substituting \( M \) by \( A \). Note that block-averaging reduces the span of delays that can be observed from the cross spectral density. Due to the averaging, the span of observable TDOAs is reduced from \( |\Delta| \leq N - 1 \) to \( |\Delta| \leq L - 1 \).

Now, using the phase curve of the block-averaged cross spectral density in (3.22) with \( N \) replaced by the block length \( L \) gives

\[
\hat{\Delta} = \frac{3}{2\pi L (L - 1)} \sum_{k=1-L}^{L-1} k \hat{\Gamma}_{avg}[k] \quad (3.46)
\]

where the phase of the averaged cross spectral density is given by

\[
\hat{\Gamma}_{avg}[k] = \angle \hat{\Phi}_{avg}[k] \quad k = 1 - L, ..., L - 1. \quad (3.47)
\]

In [18] the variance of the estimated TDOA is presented in terms of the estimated cross spectral density variance. The following analysis follows [18] closely but is adjusted for the considered discrete complex-valued model and the block-averaging. Following (3.35)-(3.43) with \( N \) replaced by \( L \), the variance of the estimator in (3.46) is

\[
\text{var}(\hat{\Delta}) = \frac{9}{4\pi^2 L^2 (L - 1)^2} \var \left( \sum_{k=1-L}^{L-1} k \hat{\Gamma}_{avg}[k] \right). \quad (3.48)
\]

The small error variance reduction in the uncorrelated bins of the block-averaged cross spectral density are reduced by a factor \( B \) and hence the
Chapter 3. Time- and frequency-domain TDOA estimation

The variance of the slope of the block-cross spectral density is also reduced by a factor $B$

$$\text{var}(\hat{\Gamma}_{\text{avg}}[k]) = \frac{\text{var}(\hat{\Gamma}_b[k])}{B} \quad k = 1 - L, ..., L - 1. \quad (3.49)$$

The variance of (3.46) is then given by

$$\text{var}(\hat{\Delta}) = \frac{9}{4\pi^2NL(L-1)^2} \sum_{k=1-L}^{L-1} k^2 \text{var}(\hat{\Gamma}_b[k]). \quad (3.50)$$

In order to analyze the effects of the block-averaging in terms of variance, the white example signal (2.21) is used. The variance for each of the phase curves is given by (3.43) with $N$ replaced by the block length $L$

$$\text{var}(\hat{\Gamma}_b[k]) \approx \left(\frac{L-|\Delta|}{L}\right)^{-2} \left[\frac{2\text{SNR} + 1}{\text{SNR}^2} + 1 - \left(\frac{L-|\Delta|}{L}\right)^2\right] \quad (3.51)$$

which inserted into (3.50) gives the variance, for the white example signal (2.21), of the frequency-domain TDOA estimator with block-averaging

$$\text{var}(\hat{\Delta}) \approx \frac{3}{4\pi^2N(L-1)} \left(\frac{L-|\Delta|}{L}\right)^{-2} \left[\frac{2\text{SNR} + 1}{\text{SNR}^2} + 1 - \left(\frac{L-|\Delta|}{L}\right)^2\right]. \quad (3.52)$$

The performance in terms of variance of the considered block-averaged frequency-domain TDOA estimator is for large $N$ and $L$ with small $\Delta$ close to the CRLB (3.10). However, for a not so large acquisition interval one may be led to believe that the smallest variance is obtained by choosing $L = N$ ($B = 1$) which is not true since (3.52) only describes the small error variance. By choosing a small $B$ the estimation of $\hat{\Gamma}_{\text{avg}}[k]$ suffers from unwrapping problems causing large errors in the TDOA estimation. These problems are suppressed by using a higher degree of averaging, that is a larger $B$ which leads to a lower variance of $\hat{\Gamma}_{\text{avg}}[k]$. In Figure 3.5 the effects of averaging are seen for a signal with similar characteristics as the white example signal (2.21). The figure shows the phase of the estimated cross spectral density using $N = 1024$ samples with no averaging ($B = 1$) and with averaging ($B = 16$). The effects of failed unwrapping are seen as steps of $2\pi$ while unwrapping of the averaged curve shows a linear phase. Failed unwrapping will result in large TDOA estimation errors since the straight line will be fitted to a non-linear phase curve.
3.4. Effects of time reference errors

Figure 3.5: The phase of an estimated cross spectral density for a signal received at 10 dB SNR using $N = 1024$ samples with no averaging $B = 1$ and averaging using $B = 16$ are presented. The failed unwrapping is seen as $2\pi$-steps.

In Figure 3.6, the effects on the mean square error (MSE) using block-averaging for the white example signal are seen. The large errors for low degrees of averaging are due to the problems with the unwrap operator. These problems are suppressed when using higher degrees of averaging since the variance of the phase is reduced. However, for really high degrees of averaging the variance will increase according to (3.52).

3.4 Effects of time reference errors

In a TDOA-based electronic warfare direction-finding system the TDOA between the outputs of two spatially separated receivers is calculated. When there is an error in the time reference signal between the two receiver systems, a biased estimate is produced. This bias is directly linked to the time-difference, or error, between the time reference outputs of the
two reference modules presented in Chapter 2.7.3.

The effects of a timing error are primarily dependent on three factors – the distance between the receivers, the direction of arrival and the propagating speed of the signal. For any given timing error and propagating speed, the directional error is minimized by placing the receivers as far apart as possible. That is, the distance between the receivers gives the maximum TDOA. For a short receiver distance the timing error causes a larger directional error since the ratio between the timing error and the maximum TDOA is large.

In a practical electronic warfare scenario, where the receivers are positioned several kilometers apart, a timing error smaller than 100 ns is sufficient in most cases. This time accuracy is achieved using the NAVSTAR-GPS system. However, the effect of the timing error in terms of directional error is also dependent on the absolute direction of arrival of the signal. This is seen using the approximate formula (1.1) in the following example.
3.4. Effects of time reference errors

Figure 3.7: A time reference error gives rise to different directional errors depending on the direction of arrival of the signal. Here, the time reference error is $\pm 0.1 \mu s$ while the distance between the receivers is 1000 meters. The two transmitters are located at $\Delta = -0.5 \mu s$ and $\Delta = 3.2 \mu s$, respectively.

**A practical example of a time reference error** Consider a scenario where two intercept receivers are placed 1000 meters, or $3.33 \mu s$ apart, as illustrated in Figure 3.7. Two signals are received one at a time by the two receivers and the only difference between the two signals are the TDOAs which are $\Delta = -0.5 \mu s$ and $\Delta = 3.2 \mu s$, respectively. Assuming that the TDOA direction-finding system suffers from a time reference error of $\pm 0.1 \mu s$, then the actual directional error for the first signal is approximately $\pm 2^\circ$ while the error for the second signal is approximately $\pm 8^\circ$ using the approximate formula in (1.1) with propagating speed $v = 3 \cdot 10^8 \text{ m/s}$. 
3.5 Effects of receiver frequency errors

In order to digitize the received signal it needs to be transposed, or mixed, to near baseband using a superheterodyne receiver, as outlined in Chapter 2.7. The oscillators in the receivers are controlled by the reference modules and in particular their frequency reference outputs. If the frequencies of these reference signals differ, the performance of the TDOA estimator is degraded as shown in the following. The receiver oscillator frequency error $\varepsilon$ is due to the frequency difference between the frequency reference outputs of the two different reference modules. In Chapter 2.7.5 it is shown that an error, or difference, in frequency between the two receiver oscillators leads to a reduced amplitude of the CCF (2.56).

Using the model presented in (2.52)-(2.53) the estimated CCF is given by (3.11). To get an understanding of how receiver oscillator frequency errors affect this estimator its expected value is calculated

$$
E\{\hat{\phi}[m]\} = \frac{N - |m|}{N} \phi_s[m + \Delta] e^{j2\pi \mu_0 \Delta + j\pi \varepsilon (1 - m)} \frac{\text{sinc}(\varepsilon (N - |m|))}{\text{sinc}(\varepsilon)} \rho_M[m]
$$

which is recognized as (2.56). In a practical system the sinc-factor reduces the contribution from the signal while the power of the noises are unaffected by the frequency error. This leads to a reduced SNR due to the frequency error which is discussed in the following.

3.5.1 Effects of frequency errors with block-averaging

In Chapter 3.3.3, block-averaging is used to reduce the variance of the estimated quantities. When block-averaging is used in presence of a receiver oscillator frequency error the block-averaging results in an amplitude reduction of the estimated CCF, or cross spectral density, and reduces the performance of the TDOA-estimator. However, some block-averaging still is needed to reduce the effects of large estimation errors. How to choose the amount of block-averaging in presence of non-zero $\varepsilon$ is discussed in Chapter 3.6. Now, in order to reduce the variance of the estimated cross spectral density, block-averaging is used. That is, the received sequences of length $N$ samples are divided into $B$ blocks each consisting of $L$ samples. An estimate of the CCF is formed for each block, $\hat{\phi}_b[k]$, and then a block-averaged estimate of the CCF is calculated as

$$
\hat{\phi}_{\text{avg}}[m] = \frac{1}{B} \sum_{b=1}^{B} \hat{\phi}_b[m].
$$

(3.54)
3.5. Effects of receiver frequency errors

Figure 3.8: The received sequences are divided into blocks of length $L$ samples each.

The individual block-CCFs are estimated as

$$\hat{\phi}_b [m] = \frac{1}{L} \sum_{n=1-N}^{N-1} r_{1b} [n + m] r_{2b} [n]$$

(3.55)

where

$$r_{1b} [n] = (s[n] + z_1 [n]) p_L \ n + \frac{L}{2} (B + 1) - bL$$

(3.56)

and

$$r_{2b} [n] = (s[n - \Delta] e^{-j2\pi \varepsilon n} + z_2 [n]) p_L \ n + \frac{L}{2} (B + 1) - bL.$$ 

(3.57)

The limits on $n$ in the received blocks (3.56)-(3.57) are directly given by how the received sequences are divided into blocks. In Figure 3.8 the limits on $n$ for $B = 4$ is presented.

Now, the block-averaging will not always increase the performance due to the frequency error which is seen by calculating the expected value of the block-averaged CCF

$$E \left\{ \hat{\phi}_{avg} [m] \right\} = \frac{1}{B} \sum_{b=1}^{B} E \left\{ \hat{\phi}_b [m] \right\}$$

(3.58)

which is, using (3.55)-(3.57)

$$E \left\{ \hat{\phi}_b [m] \right\} = \frac{1}{L} \phi_s [m + \Delta] \sum_{n=1-N-B+1}^{N-1-N-B} e^{j2\pi \varepsilon (n - \Delta - bL)} p_L [n + m] p_L [n].$$

(3.59)
The expected value of the block-averaged CCF estimate is then given by inserting ($3.59$) into ($3.58$)

\[
E\left\{ \hat{\phi}_{\text{avg}} \right\} = \frac{1}{N} \phi_d[m+\Delta] e^{-j2\pi\frac{m}{L}(B+1)} \sum_{b=1}^{B} e^{-j2\pi\varepsilon\frac{L}{2}-bL} \sum_{n=1-N/2}^{3N/2-1} e^{j2\pi\varepsilon n} p_L[n+m]p_L[n]. 
\]

The last summation is rewritten as

\[
\sum_{n=1-N/2}^{3N/2-1} e^{j2\pi\varepsilon n} p_L[n+m]p_L[n] = \sum_{n=-L/2}^{L/2} e^{j2\pi\varepsilon n} p_L[n+m] = (L-|m|) e^{j\pi\varepsilon(1-m)} \frac{\text{sinc} (\varepsilon (L-|m|))}{\text{sinc} (\varepsilon)} \hat{p}_A[m].
\]

where the second equality follows from Appendix A.1. The summation over the block-averaging is, using Appendix A.2

\[
\frac{1}{B} \sum_{b=1}^{B} e^{-j2\pi\varepsilon\frac{L}{2}-bL} = e^{-j\pi\varepsilon L(B+1)} \frac{\text{sinc} (\varepsilon N)}{\text{sinc} (\varepsilon L)}. 
\]

Now, the expected value of the block-averaged CCF in ($3.60$) is using ($3.61$)-($3.62$) given by

\[
E\left\{ \hat{\phi}_{\text{avg}} \right\} = \frac{L-|m|}{L} \phi_d[m+\Delta] e^{j\pi\varepsilon(1-m-2L(B+1))} \frac{\text{sinc} (\varepsilon N) \text{sinc} (\varepsilon (L-|m|))}{\text{sinc} (\varepsilon) \text{sinc} (\varepsilon L)} \hat{p}_A[m].
\]

where the sinc-factors gives an additional amplitude reduction due to the block-averaging in presence of a receiver frequency error. The block-averaged CCF is used to calculate the block-averaged estimate of the cross spectral density

\[
\hat{\phi}_{\text{avg}}[k] = \mathcal{F}_A \left\{ \hat{\phi}_{\text{avg}} \right\}. 
\]

The expected value of the estimated averaged cross spectral density is given by the expected value of the estimated averaged CCF as

\[
E\left\{ \hat{\phi}_{\text{avg}}[k] \right\} = \mathcal{F}_A \left\{ E\left\{ \hat{\phi}_{\text{avg}} \right\} \right\}.
\]
3.5. Effects of receiver frequency errors

with \( A = 2L - 1 \). For the white example signal in (2.21), the block-
averaged cross spectral density is

\[
E \left\{ \Phi_{\text{avg}} [k] \right\} = \frac{L - |\Delta|}{L} \sigma_s^2 e^{j\pi \varepsilon (1 + \Delta - 2L(B+1)) + j2\pi k \Delta / A} \frac{\text{sinc} (\varepsilon N)}{\text{sinc} (\varepsilon)} \frac{\text{sinc} (\varepsilon (L - |\Delta|))}{\text{sinc} (\varepsilon L)}
\]

(3.66)

for \( k = 1 - L, \ldots, L - 1 \). The (expected) slope of the cross spectral density
in (3.66) used in the TDOA estimator (3.22) is then for the white example
signal given by

\[
\angle E \left\{ \Phi_{\text{avg}} [k] \right\} = 2\pi k \Delta / A + \pi \varepsilon (1 + \Delta - 2L (B + 1)) \quad k = 1 - L, \ldots, L - 1
\]

(3.67)

where it is seen that a frequency error in combination with block-averaging
only affects the bias of the slope. However, the frequency error in com-
bination with block-averaging yield an additional amplitude degradation
(3.63) which results in a performance degradation of the TDOA estimator.

Numerical simulations based on sampled data are run to evaluate the
performance in presence of a frequency error. The presented results are
based on 1000 independent Monte-Carlo simulations. Considering crystal
oscillators or NAVSTAR-GPS disciplined Rubidium oscillators to be used
with the intercept receivers, then the typical relative frequency errors
are \( 10^{-5} \) and \( 10^{-10} \), respectively. In this example the assumed maxi-
mum receiver mixer frequency is \( 10^{10} \) Hz which for sampling frequency
\( f_s = 10^7 \) Hz yields the interval of interest for \( \varepsilon \). That is, \( \varepsilon < 10^{-2} \) and
accordingly the numerical simulations are run with \( 10^{-7} < \varepsilon < 10^{-2} \) for a
high SNR case (20 dB) and a low SNR case (0 dB). The results, using the
white example signal, are presented in Figures 3.9 and 3.10 using differ-
ent degrees of averaging. From the figures it is evident that an ordinary
crystal oscillator with \( \varepsilon \approx 10^{-2} \) is not stable enough while a NAVSTAR-
GPS disciplined oscillator with \( \varepsilon \approx 10^{-7} \) clearly is stable enough since the
mean square error attains the CRLB (\( \varepsilon = 0 \)) for small enough \( \varepsilon \). In the
high SNR case it is seen that the CRLB is attained for small frequency
errors and a suitable degree of averaging. The low SNR case is shown in
Figure 3.10 where the CRLB is attained for a suitable choice of \( B \) but
since the SNR is lower, a higher degree of averaging is needed to attain
the CRLB.
Figure 3.9: The MSE is determined by numerical simulations using the example signal. Here, the SNR is 20 dB and $B = 32$ ($N = 1024$) gives the lowest MSE for all small frequency errors ($\varepsilon < 1/N$).
3.5. Effects of receiver frequency errors

Figure 3.10: The MSE is determined by numerical simulations using the example signal. Here, the SNR is 0 dB and $B = 128$ ($N = 1024$) gives the lowest MSE for all small frequency errors ($\varepsilon < 1/N$).
3.6 Robustness against frequency errors

In previous chapters it is shown that the performance of the TDOA estimator is degraded in presence of a frequency error between the receiver oscillators. In this chapter a power ratio $Q$ is defined which describes the power-loss due to the frequency error resulting in the TDOA-estimator performance degradation. Using the $Q$-value it is possible to calculate the power-loss in the system due to the frequency error for the given signal. The $Q$-value is defined as

$$Q = \frac{P_{out}}{P_{in}}$$

(3.68)

where $P_{out}$ is the total power of the calculated cross spectral density between the sensor outputs

$$P_{out} = \frac{1}{M} \sum_{k=-(N-1)}^{N-1} |\Phi[k]|$$

(3.69)

and $P_{in}$ is the power of the signal, that is $P_{in} = \phi_s[0]$. The power ratio $Q$ describes the relative loss of power due to the frequency error $\varepsilon$ described by (3.63) and (3.66). Ideally, the $Q$-value should be close to unity, indicating no loss in power. For example, consider the basic model in Chapter 1.2 where $\Delta_t = \Delta f_s^{-1}$

$$P_{out} = \int_{-\infty}^{\infty} |\Phi_s(f)| e^{i2\pi f \Delta} | df = \phi_s(0)$$

(3.70)

and thus $Q \equiv 1$. One may note that the $Q$-value defined in (3.68) depends on the auto-correlation function of the source signal. The $Q$-value can also be numerically evaluated for different values of the influencing parameters, that is physical conditions ($\Delta$, SNR, $\phi_s[m]$), system configuration ($N, f_s$) and system imperfections ($\varepsilon$).

In order to get some insight into the behavior of the $Q$-value, an example is considered. Given the white example signal (2.21) with a TDOA described by $\Delta$, the output power $P_{out}$ is given by inserting the cross spectral density in (2.57) into (3.69). A straightforward calculation gives

$$P_{out} = \frac{N - |\Delta|}{N} \left| \frac{\text{sinc}(\varepsilon(N - |\Delta|))}{\text{sinc}(\varepsilon)} \right| \sigma_s^2$$

(3.71)

while the input power is

$$P_{in} = \sigma_s^2.$$  

(3.72)
3.6. Robustness against frequency errors

The resulting $Q$-value becomes

$$Q = \frac{N - |\Delta|}{N} \left| \frac{sinc(\varepsilon(N - |\Delta|))}{sinc(\varepsilon)} \right|$$

and, clearly, it depends on $\varepsilon, \Delta$ and $N$. In Figure 3.11, the $Q$-value (3.73) is shown for the white example signal with $N = 1024$ and $\Delta = 10$ samples. From (3.73) one may note that $Q \to (N - |\Delta|)/N \approx 1$ when $\varepsilon \to 0$ as expected. Also note that for a fixed $\varepsilon$, $Q \to 0$ as $N \to \infty$ due to the decorrelation between the channels from the frequency error.

However, to better describe the effects of the frequency error in a practical scenario, the $Q$-value needs to be estimated from the received sequences. This estimated $Q$-value is now defined as

$$\hat{Q} = \frac{\hat{P}_{out}}{\hat{P}_{in}}$$

where the estimated output power is calculated using the estimated cross
spectral density
\[
\hat{P}_{out} = \frac{1}{M} \sum_{k=1-N}^{N-1} \left| \hat{\Phi} [k] \right|
\]  
(3.75)
and \( \hat{P}_{in} \) is an estimate of the signal power \( P_{in} \). As illustrated in Figure 3.11 the estimated \( Q \) attains the modelled \( Q \) for small frequency errors. However, for large frequency errors (\( \varepsilon > 1/N \)) the modelled and the estimated \( Q \)-values differ. To illustrate these effects the white example signal (2.21) with \( \Delta = 0 \) in a noise-free scenario is used. The estimated cross spectral density in (3.75) is given by (3.23)
\[
\hat{\Phi} [k] = \frac{1}{N} R_1 [k] R_2^* [k] \quad k = 1 - N, ..., N - 1
\]  
(3.76)
where \( R_1 [k] = \mathcal{F}_M \{ r_1 (n) \} \) and \( R_2 [k] = \mathcal{F}_M \{ r_2 (n) \} \). The expected value of the estimated output power is calculated to get some insight into the behavior of the estimated \( Q \)-value
\[
E \{ \hat{P}_{out} \} = \frac{1}{M} \sum_{k=1-N}^{N-1} E \{ |\hat{\Phi} [k]| \}
\]  
(3.77)
\[
= \frac{1}{MN} \sum_{k=1-N}^{N-1} E \{ |R_1 [k]| R_2^* [k]| \}.
\]  
(3.78)
The large frequency error decorrelates the DFTs of the received sequences which leads to
\[
E \{ \hat{P}_{out} \} = \frac{1}{MN} \sum_{k=1-N}^{N-1} E \{ |R_1 [k]| \} E \{ |R_2^* [k]| \}. 
\]  
(3.79)
Calculating the DFT of the white example signal gives a circular, complex-valued Gaussian sequence with variance \( N \sigma_s^2 \). The expectations in (3.78) is then, for a noise free analysis, calculated using Appendix B.1
\[
E \{ |R_1 [k]| \} = E \{ |R_2 [k]| \} = \sqrt{\frac{N \pi \sigma_s^2}{4}}.
\]  
(3.79)
By inserting (3.79) into (3.78), a straightforward calculation gives
\[
E \{ \hat{P}_{out} \} = \frac{\pi \sigma_s^2}{4}
\]  
(3.80)
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with the input power given by $\sigma_n^2$. The (expected) estimated $Q$-value for the white example signal suffering from large frequency errors is then given by

$$E \left\{ \hat{Q} \right\} = \frac{E \left\{ \hat{P}_{\text{out}} \right\}}{\sigma_n^2} = \frac{\pi}{4} \approx 0.785. \quad (3.81)$$

This means that the estimated $Q$-value for a white noise signal always is between unity and $\pi/4$. When $\hat{Q}$ attains the threshold at $\pi/4$ the frequency error is large, $\varepsilon > 1/N$, and the TDOA-estimator fails to produce a reliable estimate due to the decorrelation between the received sequences.

Block-averaging of the cross spectral density will change both the range and the level of the threshold since the blocklength determines at which $\varepsilon$ the TDOA estimator fails to produce a reliable result, that is the estimated $Q$-value reaches the threshold level for large frequency errors given by $\varepsilon > 1/L$. In the following, the effects of block-averaging on the estimated $Q$-value are discussed. The estimated output power is now calculated using the block-averaged cross spectral density

$$\hat{\Phi}_{\text{avg}}[k] = \frac{1}{B} \sum_{b=1}^{B} \hat{\Phi}_b[k] \quad k = 1 - L, \ldots, L - 1 \quad (3.82)$$

where the block-cross spectral densities $\hat{\Phi}_b[k]$ are calculated using the block DFTs $R_{1b}[k]$ and $R_{2b}[k]$

$$\hat{\Phi}_b[k] = \frac{1}{L} R_{1b}[k] R_{2b}^*[k] \quad k = 1 - L, \ldots, L - 1. \quad (3.83)$$

The estimate of the output power is then

$$\hat{P}_{\text{out}} = \frac{1}{A} \sum_{k=1-L}^{L-1} \left| \hat{\Phi}_{\text{avg}}[k] \right| \quad (3.84)$$

where $A = 2L - 1$. To get some understanding of the behavior of the output power, the expected value is calculated using the white example signal (2.21). The expected value of (3.84) is then given by

$$E \left\{ \hat{P}_{\text{out}} \right\} = \frac{1}{A} \sum_{k=1-L}^{L-1} E \left\{ \left| \hat{\Phi}_{\text{avg}}[k] \right| \right\} \quad (3.85)$$

which, using (3.66) with $\hat{P}_{\text{in}} = \sigma_n^2$ gives the expected value of $\hat{Q}$ using block-averaging

$$E \left\{ \hat{Q} \right\} = \frac{L - |\Delta|}{L} \frac{\text{sinc} \left( \varepsilon N \right)}{\text{sinc} \left( \varepsilon \right)} \left| \frac{\text{sinc} \left( \varepsilon \left( L - |\Delta| \right) \right)}{\text{sinc} \left( \varepsilon L \right)} \right|. \quad (3.86)$$
Note, that for no averaging with $B = 1$ ($L = N$) the expression in (3.86) becomes (3.73).

For large frequency errors ($\varepsilon > 1/L$) the analysis above will not hold since $R_{1b}[k]$ and $R_{2b}^*[k]$ in (3.83) becomes uncorrelated due to the large frequency error. Now, $E\{\hat{\Phi}_{\text{avg}}[k]\}$ in (3.85) is calculated using the definition of the variance as

$$E\left\{ \left| \hat{\Phi}_{\text{avg}}[k] \right| \right\} = \sqrt{E\left\{ \left| \hat{\Phi}_{\text{avg}}[k] \right|^2 \right\} - \text{var}\left\{ \left| \hat{\Phi}_{\text{avg}}[k] \right| \right\}. \quad (3.87)$$

Note that

$$\text{var}\left\{ \left| \hat{\Phi}_{\text{avg}}[k] \right| \right\} = \frac{1}{B} \text{var}\left\{ \left| \hat{\Phi}_b[k] \right| \right\} \quad (3.88)$$

and

$$E\left\{ \left| \hat{\Phi}_{\text{avg}}[k] \right|^2 \right\} = \text{var}\left\{ \hat{\Phi}_{\text{avg}}[k] \right\} = \frac{1}{B} \text{var}\left\{ \hat{\Phi}_b[k] \right\} \quad (3.89)$$

since the blocks are uncorrelated and the white example signal is zero-mean. Inserting (3.88) and (3.89) into (3.87) gives

$$E\left\{ \left| \hat{\Phi}_{\text{avg}}[k] \right| \right\} = \sqrt{\frac{1}{B} \left( \text{var}\left\{ \hat{\Phi}_b[k] \right\} - \text{var}\left\{ \left| \hat{\Phi}_b[k] \right| \right\} \right)}. \quad (3.90)$$

Again, use the definition of the variance to obtain

$$\text{var}\left\{ \hat{\Phi}_b[k] \right\} = E\left\{ \left| \hat{\Phi}_b[k] \right|^2 \right\} - E^2\left\{ \hat{\Phi}_b[k] \right\} \quad (3.91)$$

and

$$\text{var}\left\{ \left| \hat{\Phi}_b[k] \right| \right\} = E\left\{ \left| \hat{\Phi}_b[k] \right|^2 \right\} - E^2\left\{ \left| \hat{\Phi}_b[k] \right| \right\} \quad (3.92)$$

which inserted into (3.90) gives

$$E\left\{ \left| \hat{\Phi}_{\text{avg}}[k] \right| \right\} = \sqrt{\frac{1}{B} \left( \text{var}\left\{ \hat{\Phi}_b[k] \right\} - \text{var}\left\{ \left| \hat{\Phi}_b[k] \right| \right\} \right)}. \quad (3.93)$$

The block DFTs are uncorrelated for large frequency errors ($\varepsilon > 1/L$) and the white example signal is zero-mean, which gives

$$E\left\{ \hat{\Phi}_b[k] \right\} = \frac{1}{A} E\left\{ R_{1b}[k] R_{2b}^*[k] \right\} = \frac{1}{A} E\left\{ R_{1b}[k] \right\} E\left\{ R_{2b}^*[k] \right\} = 0. \quad (3.94)$$
3.6. Robustness against frequency errors

Now, (3.93) is evaluated using (3.83), (3.94) and Appendix B.1

\[
E \left\{ |\hat{\Phi}_{avg}[k]| \right\} = \frac{1}{\sqrt{B}} E \left\{ |\hat{\Phi}_b[k]| \right\} = \frac{\pi \sigma_s^2}{4\sqrt{B}}
\]  
(3.95)

and the expected value of the output power in (3.85) is

\[
E \left\{ \hat{P}_{out} \right\} = \frac{\pi \sigma_s^2}{4\sqrt{B}}.
\]  
(3.96)

The estimated \( Q \)-value is for the white example signal with block-averaging then given by

\[
E \left\{ \hat{Q} \right\} \approx \frac{\pi}{4\sqrt{B}}.
\]  
(3.97)

In Figure 3.12 it is seen that block-averaging of the cross spectral density helps reduce the threshold effect following the large frequency errors for a noise-free scenario. However, the degree of block-averaging \( B \) is to be chosen with respect to both the frequency error and the SNR as shown in Chapter 3.3.3 to obtain an unbiased TDOA estimate with low variance. Note that the ripple for large \( \varepsilon \) in Figure 3.12 also is visible in Figures 3.9 and 3.10. Also note that for a very high degree of averaging (here \( B = 256 \)) the amplitude is reduced since the ratio

\[
\frac{L - |\Delta|}{L}
\]  
(3.98)

becomes small in (3.63).
Figure 3.12: The $Q$-value for the example signal with $N = 2^{10}$ and $\Delta = 1$ with different degrees of block-averaging.
Chapter 4

Conclusions and topics for future work

In this thesis both time- and frequency-domain based TDOA estimators are derived and the performance in terms of bias and variance is calculated using the white example signal in (2.21). Numerical simulations based on sampled data are used to evaluate the estimator performance and is compared to the CRLB in Chapter 3.1. It is shown that the considered estimators have similar performance and that the CRLB is attained for large data records. In the considered electronic warfare scenario the SNR is assumed low and block-averaging of the spectral estimators are used to reduce the variance of the TDOA estimators.

In a practical system, reference imperfections will degrade the performance of a TDOA-based direction-finding system. In particular the effects of time and frequency reference errors are studied. It is shown that a time error gives a biased TDOA estimate while a frequency error results in a noise like degradation due to an amplitude reduction in the CCF (2.56). Moreover, block-averaging will increase the amplitude reduction when a frequency error exist (3.63), that is the effects of a frequency error will be more serious if block-averaging is used. In Chapter 3.6 the robustness against a frequency error for a specific scenario is calculated using the $Q$-value which describes the amplitude reduction due to a frequency error.
The next natural step is to estimate the frequency error from the received sequences. A method of joint estimation of the TDOA and the frequency error is presented in [21] and [23]. The performance degradation due to the frequency error is then reduced and the variance of the TDOA estimator is reduced so that the CRLB is (asymptotically) attained.

In the considered electronic warfare scenario with low SNR, block-averaging is used to reduce the variance of the TDOA estimates to obtain useful results. However, for any given scenario, regardless of any reference errors, the parameters used in the block-averaging must be chosen with respect to the actual scenario. If chosen correct, the TDOA estimator variance is minimized for that particular scenario. However, the optimum choice of the block-averaging parameters is not discussed in this thesis and is left for future work.

Several system models considering TDOA-estimation with receiver reference imperfections are derived and analyzed in this thesis. However, no multipath scenarios are considered and hence no system models containing the effects of multipath are derived. An interesting path for future work would include system models for multipath scenarios where a combination of the auto-correlation functions and the cross-correlation function can be used to identify direct and indirect paths for increased accuracy in the TDOA estimation process.
Appendix A

Useful sums

A.1 Time averaging of frequency error

When analyzing the effects of limited acquisition intervals in combination with a frequency error the CCF needs to be time-averaged. This time-averaging gives rise to a summation over a complex rotation and a rectangular pulse as shown below. The result is a complex rotation and an amplitude reduction due to the frequency error.

Claim:

\[
\frac{N}{2} \sum_{n=-N/2+1}^{N/2} e^{j2\pi n} p_N[n + m] = (N - |m|) e^{j\pi \varepsilon (1-m)} \frac{\text{sinc} (\varepsilon (N - |m|))}{\text{sinc} (\varepsilon)} p_M'[m]
\]  

(A.1)

Proof:

First, identify the two separate cases for positive or negative \( m \) given by the properties of the rectangular window

\[
\sum_{n=-N/2+1}^{N/2} e^{j2\pi n} p_N[n + m] = \begin{cases} 
\sum_{n=-N/2+1}^{N/2-m} e^{j2\pi n} p_M'[m] & m \geq 0 \\
\sum_{n=-N/2+1-m}^{N/2} e^{j2\pi n} p_M'[m] & m < 0 
\end{cases}
\]

(A.2)

For negative \( m \), rewrite the summation limits to comply with standard
summation formulas

\[
\sum_{n=-N/2+1}^{N/2} e^{j2\pi \varepsilon n} = e^{j2\pi \varepsilon (1-N/2-m)} \sum_{n=0}^{N-1+m} e^{j2\pi \varepsilon n} \tag{A.3}
\]

where the last sum is evaluated using Euler's formula

\[
\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \tag{A.4}
\]

That is,

\[
\sum_{n=0}^{N-1+m} e^{j2\pi \varepsilon n} = \frac{\sin(\pi \varepsilon (N + m))}{\sin(\pi \varepsilon)} e^{j\pi \varepsilon}. \tag{A.5}
\]

Inserting (A.5) into (A.3) gives

\[
\sum_{n=-N/2+1}^{N/2-m} e^{j2\pi \varepsilon n} = e^{j2\pi \varepsilon (1-N/2-m)} \frac{\sin(\pi \varepsilon (N + m))}{\sin(\pi \varepsilon)} e^{j\pi \varepsilon} = e^{j\pi \varepsilon (1-m)} \frac{\sin(\pi \varepsilon (N + m))}{\sin(\pi \varepsilon)}. \tag{A.6}
\]

For \(m \geq 0\)

\[
\sum_{n=-N/2+1}^{N/2-m} e^{j2\pi \varepsilon n} = e^{j2\pi \varepsilon (1-N/2)} \sum_{n=0}^{N-1-m} e^{j2\pi \varepsilon n} \tag{A.7}
\]

where the last summation is evaluated to

\[
\sum_{n=0}^{N-1-m} e^{j2\pi \varepsilon n} = \frac{\sin(\pi \varepsilon (N - m))}{\sin(\pi \varepsilon)} e^{j\pi \varepsilon}. \tag{A.8}
\]

Inserting (A.8) into (A.7) gives

\[
\sum_{n=-N/2+1}^{N/2-m} e^{j2\pi \varepsilon n} = e^{j2\pi \varepsilon (1-N/2)} \frac{\sin(\pi \varepsilon (N - m))}{\sin(\pi \varepsilon)} e^{j\pi \varepsilon} = e^{j\pi \varepsilon (1-m)} \frac{\sin(\pi \varepsilon (N - m))}{\sin(\pi \varepsilon)}. \tag{A.9}
\]
Now, it is seen that (A.6) and (A.9) are equal in most parts and a formula for all $m$ is achieved using

$$\sum_{n=-N/2+1}^{N/2} e^{j2\pi \epsilon n} p_N[n + m] = e^{j\pi \epsilon (1-m)} \frac{\sin (\pi \epsilon (N - |m|))}{\sin (\pi \epsilon)} p_M'[m]$$

$$= (N - |m|) e^{j\pi \epsilon (1-m)} \frac{\sin (\epsilon (N - |m|))}{\sin (\epsilon)} p_M'[m]$$

(A.10)

where

$$\text{sinc} (x) = \frac{\sin (\pi x)}{\pi x}.$$  \hspace{1cm} (A.11)

Note that

$$\frac{\sin (\epsilon (N - |m|))}{\sin (\epsilon)} = 1 \text{ for } |\epsilon| = 0, 1, 2, ...$$  \hspace{1cm} (A.12)

and

$$\frac{\sin (\epsilon (N - |m|))}{\sin (\epsilon)} = 0 \text{ for } |\epsilon (N - |m|)| = 1, 2, ... \text{ and } 0 < |\epsilon| < 1.$$  \hspace{1cm} (A.13)

The effects of the amplitude loss due to limited acquisition interval in combination with a frequency error are shown in Figure A.1 where it is seen that the amplitude loss depends on the acquisition interval $N$ as well as the frequency error $\epsilon$. 
Figure A.1: The amplitude loss due to $\varepsilon$ varies with $N$. Large $N$ gives a higher amplitude loss since $1/N$ is smaller.
A.2 Block-averaging of frequency error

When analyzing block-averaging in combination with a frequency error a summation over a complex rotation arises. The summation is calculated over \( B \) blocks each of length \( L \) samples for a total of \( N \) samples. The result is a complex rotation and an amplitude reduction due to the block-averaging.

Claim:

\[
\frac{1}{B} \sum_{b=1}^{B} e^{-j2\pi \varepsilon_b L} = e^{-j\pi \varepsilon L (B+1)} \frac{sinc(\varepsilon N)}{sinc(\varepsilon L)} \tag{A.14}
\]

Proof:

First, let \( \alpha = e^{-j2\pi \varepsilon L} \) for notational simplicity, then

\[
\frac{1}{B} \sum_{b=1}^{B} e^{-j2\pi \varepsilon_b L} = \frac{1}{B} \sum_{b=1}^{B} \alpha^b = \frac{\alpha}{B} \sum_{b=0}^{B-1} \alpha^b \tag{A.15}
\]

where the last sum is a standard sum

\[
\frac{\alpha}{B} \sum_{b=0}^{B-1} \alpha^b = \frac{\alpha - \alpha^B}{1 - \alpha} = \frac{\alpha^{(B-1)/2} - \alpha^{B/2}}{\alpha^{1/2} - \alpha^{1/2}} \tag{A.16}
\]

Use Euler’s formula on (A.16) with \( \alpha = e^{-j2\pi \varepsilon L} \) and \( N = BL \)

\[
\sum_{b=1}^{B} e^{-j2\pi \varepsilon_b L} = e^{-j\pi \varepsilon L (B+1)} \frac{e^{j\pi \varepsilon N} - e^{-j\pi \varepsilon N}}{e^{j\pi \varepsilon L} - e^{-j\pi \varepsilon L}} = e^{-j\pi \varepsilon L (B+1)} \frac{\sin(\pi \varepsilon N)}{\sin(\pi \varepsilon L)} \tag{A.17}
\]

and (A.15) is for \( \text{sinc}(x) = \sin(\pi x) / \pi x \) evaluated to

\[
\frac{1}{B} \sum_{b=1}^{B} e^{-j2\pi \varepsilon_b L} = e^{-j\pi \varepsilon L (B+1)} \frac{sinc(\varepsilon N)}{sinc(\varepsilon L)} \tag{A.18}
\]

Note the amplitude variations for the sinc-factors with extremes at

\[
\frac{sinc(\varepsilon N)}{sinc(\varepsilon L)} = 1 \text{ for } |\varepsilon L| = 0, 1, 2, ... \tag{A.19}
\]

and

\[
\frac{sinc(\varepsilon N)}{sinc(\varepsilon L)} = 0 \text{ for } |\varepsilon N| = 1, 2, ... \tag{A.20}
\]

since \( N = BL \).
A.3 Sum of powers

Three standard sums of powers are used:

Claim [20]:

\[ \sum_{k=0}^{N-1} k = \frac{N(N-1)}{2} \]  \hspace{1cm} (A.21)

Claim [20]:

\[ \sum_{k=0}^{N-1} k^2 = \frac{N(N-1)(2N-1)}{6} \]  \hspace{1cm} (A.22)

Claim [20]:

\[ \sum_{k=0}^{N-1} \alpha^k = \frac{1 - \alpha^N}{1 - \alpha} \]  \hspace{1cm} (A.23)
Appendix B

Complex-valued random variables

In this thesis complex-valued random variables are used and are defined below. This definition follows closely the definition in [13]. Let $z(t)$ be a complex-valued random variable defined as

$$z(t) = u(t) + jv(t) \quad (B.1)$$

where $u(t)$ and $v(t)$ are real-valued, mutually uncorrelated and identically distributed random variables. Let

$$E\{u(t)\} = E\{v(t)\} = \bar{u} \quad (B.2)$$

and

$$E\{u^2(t)\} = E\{v^2(t)\} = \sigma_z^2/2 + \bar{u}^2 \quad (B.3)$$

with

$$\text{var}\{u(t)\} = \text{var}\{v(t)\} = \sigma_z^2/2. \quad (B.4)$$

Then the complex-valued random variable $z(t)$ have the following properties

$$\bar{z} = E\{z(t)\} = E\{u(t)\} + jE\{v(t)\} = \bar{u} (1 + j) \quad (B.5)$$

and

$$E\{|z(t)|^2\} = E\{u^2(t)\} + E\{v^2(t)\} = 2 \left(\sigma_z^2/2 + \bar{u}^2\right). \quad (B.6)$$
The variance of the complex-valued random variable follows from

\[
\text{var} \{ z(t) \} = \mathbb{E} \left\{ |z(t) - \mathbb{E} \{ z(t) \}|^2 \right\} \\
= \mathbb{E} \left\{ |z(t)|^2 \right\} - |\mathbb{E} \{ z(t) \}|^2
\]  

(B.7)

which is evaluated to

\[
\text{var} \{ z(t) \} = 2\mathbb{E} \{ u^2(t) \} - 2\mathbb{E}^2 \{ u(t) \} \\
= 2\text{var} \{ u(t) \} \\
= \sigma_z^2.
\]  

(B.8)

For a complex valued Gaussian variable \( z(t) = u(t) + jv(t) \) where \( u \) and \( v \) are independent and identically distributed denoted \( u, v \in \mathcal{N}(\bar{u}, \sigma^2) \) the probability density function (PDF) is given by the joint PDF of \( u, v \)

\[
p(u, v) = \frac{1}{\sqrt{\pi \sigma_z^2}} \exp \left\{ -\frac{1}{\sigma_z^2} (u - \bar{u})^2 \right\} \cdot \frac{1}{\sqrt{\pi \sigma_z^2}} \exp \left\{ -\frac{1}{\sigma_z^2} (v - \bar{v})^2 \right\} \\
= \frac{1}{\pi \sigma_z^2} \exp \left\{ -\frac{1}{\sigma_z^2} \left( (u - \bar{u})^2 + (v - \bar{v})^2 \right) \right\}.
\]  

(B.9)

The PDF of the complex-valued Gaussian variable \( z(t) \) is then

\[
p(z) = \frac{1}{\pi \sigma_z^2} \exp \left\{ -\frac{1}{\sigma_z^2} |z - \bar{z}|^2 \right\}
\]  

(B.10)

which is denoted

\[ z \sim \mathcal{CN}(\bar{z}, \sigma_z^2). \]  

(B.11)
B.1 The expected value of the magnitude of a complex-valued Gaussian variable

B.1 The expected value of the magnitude of a complex-valued Gaussian variable

Claim:

The expected value of the magnitude of a complex-valued zero mean Gaussian variable $z$ with variance $\sigma_z^2$ is given by

$$ E\{|z|\} = \sqrt{\frac{\pi \sigma_z^2}{4}} \tag{B.12} $$

Proof:

A complex-valued Gaussian random variable $z \sim \mathcal{CN}(\bar{z}, \sigma_z^2)$ with a PDF according to (B.10) is used to calculate the expected value from the definition

$$ E\{|z|\} = E\{|u + jv|\} = \frac{1}{\pi \sigma_z^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u + jv| \exp \left( -\frac{1}{\sigma_z^2} (u^2 + v^2) \right) dudv. \tag{B.13} $$

The integral is evaluated with a transformation of $z$ into polar coordinates

$$ r = |z| = \sqrt{u^2 + v^2} \tag{B.14} $$
$$ \theta = \tan^{-1} \left( \frac{v}{u} \right) \tag{B.15} $$
$$ dudv = rdrd\theta \tag{B.16} $$

which inserted into (B.13) gives

$$ E\{|z|\} = \frac{1}{\pi \sigma_z^2} \int_0^{\infty} \int_0^{2\pi} r^2 \exp \left( -\frac{1}{\sigma_z^2} r^2 \right) drd\theta = \frac{2}{\sigma_z^2} \int_0^{\infty} r^2 \exp \left( -\frac{1}{\sigma_z^2} r^2 \right) dr. \tag{B.17} $$

The solution of the remaining integral is found in a table of integrals [20]

$$ E\{|z|\} = \frac{2 \sigma_z^2}{\sigma_z^2} \frac{\sqrt{\sigma_z^2 \pi}}{4} = \sqrt{\frac{\sigma_z^2 \pi}{4}}. \tag{B.18} $$
Bibliography


