Some aspects of optimal switching and pricing
Bermudan options

ALI HAMDI

Doctoral Thesis
Stockholm, Sweden 2013
Akademisk avhandling som med tillstånd av Kungl Tekniska högskolan framlägges till offentlig granskning för avläggsning av teknologie doktorsexamen fredagen den 17 maj 2013 klockan 10.00 i sal F3, Kungl Tekniska högskolan, Lindstedtsvägen 26, Stockholm.

© Ali Hamdi, 2013

Tryck: Universitetsservice US AB
Abstract

This thesis consists of four papers that are all related to the Snell envelope. In the first paper, the Snell envelope is used as a formulation of a two-modes optimal switching problem. The obstacles are interconnected, take both profit and cost yields into account, and switching is based on both sides of the balance sheet. The main result is a proof of existence of a continuous minimal solution to a system of Snell envelopes, which fully characterizes the optimal switching strategy. A counter-example is provided to show that uniqueness does not hold.

The second paper considers the problem of having a large number of production lines with two modes of production, high-production and low-production. As in the first paper, we consider both expected profit and cost yields and switching based on both sides of the balance sheet. The production lines are assumed to be interconnected through a coupling term, which is the average optimal expected yields. The corresponding system of Snell envelopes is highly complex, so we consider the aggregated yields where a mean-field approximation is used for the coupling term. The main result is a proof of existence of a continuous minimal solution to a system of Snell envelopes, which fully characterizes the optimal switching strategy. Furthermore, existence and uniqueness is proven for the mean-field reflected backward stochastic differential equations (MF-RBSDEs) we consider, a comparison theorem and a uniform bound for the MF-RBSDEs is provided.

The third paper concerns pricing of Bermudan type options. The Snell envelope is used as a representation of the price, which is determined using Monte Carlo simulation combined with the dynamic programming principle. For this approach, it is necessary to estimate the conditional expectation of the future optimally exercised payoff. We formulate a projection on a grid which is ill-posed due to overfitting, and regularize with the PDE which characterizes the underlying process. The method is illustrated with numerical examples, where accurate results are demonstrated in one dimension.

In the fourth paper, the idea of the third paper is extended to the multi-dimensional setting. This is necessary because in one dimension it is more efficient to solve the PDE than to use Monte Carlo simulation. We relax the use of a grid in the projection, and add local weights for stability. Using the multi-dimensional Black-Scholes model, the method is illustrated in settings ranging from one to 30 dimensions. The method is shown to produce accurate results in all examples, given a good choice of the regularization parameter.
Acknowledgments

I wish to thank my supervisor Professor Boualem Djehiche and my co-supervisor Associate Professor Henrik Hult for their guidance, their support and encouragement, and for many inspiring discussions throughout my time as a student. Furthermore, I want to thank Associate Professor Filip Lindskog for discussions and comments regarding the third paper, Pierre Nyquist for discussions and comments regarding the fourth paper, and Thorbjörn Gudmundsson for comments regarding the introduction.

The financial support from the Swedish Export Credit Corporation (SEK) is gratefully acknowledged. In addition, I want to thank Per Åkerlind, Richard Anund, Jonas Söderberg, Erik Brodin and Gustav Hast for making this possible and for their continuous support and encouragement.
Contents

Introduction to financial derivatives .......................... 1
An overview of arbitrage-free pricing theory ................ 2
The Snell envelope and its connection to RBSDEs .......... 7
An overview of optimal switching ............................ 9

Summary of the papers ........................................ 11
Paper I .................................................................. 11
Paper II ............................................................... 12
Paper III ............................................................. 14
Paper IV ............................................................. 16

Bibliography ......................................................... 20

List of papers


Introduction

In this section we give a brief introduction to financial derivatives and their valuation using arbitrage-free pricing theory. A specific type of derivative, Bermudan callable, will be discussed and we will mention some specific challenges in its valuation.

When presenting the pricing approach for Bermudan callables, the notion of the Snell envelope will be introduced. Some results regarding the Snell envelope and its connection to reflected backward stochastic differential equations (RBSDEs) will be presented. An overview of optimal switching is given next, where we will mention why the Snell envelope and its connection to RBSDEs is an important tool in this context. The connection between optimal switching and arbitrage-free pricing theory will be briefly highlighted. We conclude by giving a summary of the papers in this thesis.

Introduction to financial derivatives

Consider a Swedish company that relies on suppliers in the US. The base currency of the company is the Swedish krona (SEK), while it pays for most of its supplies in US dollars (USD). The company estimates that it will be needing a large amount of supplies one year from now, for which the company will have to pay exactly 1,000,000 USD. Since the base currency of the company is SEK, it will have to make a currency exchange at some point.

The company feels that, given what it charges for its product, it can pay no more than 7,000,000 SEK for the supplies. That is, it has to make the currency exchange at a rate no larger than 7 SEK for each USD. At the time of this writing the USD/SEK exchange rate is about 6.7, but the payment will not be made until delivery and there is no guarantee that the exchange rate will stay below 7.

To be sure to have the dollars ready one year from now, while securing a rate below 7, the company can make the exchange today and simply store the dollars until it is time to pay for the supplies. However, this ties up a large amount of money which could be used elsewhere during the time. In addition, if the exchange rate drops then the company wants to buy the dollars at the market price.

Another way to secure the rate is to purchase a European call option...
on the USD/SEK exchange rate for 1,000,000 USD. The formal definition of a call option is that the buyer has the right, but not the obligation, to buy the underlying asset (exercise the option) at a prespecified time and at a prespecified price. The price is called the strike price and is usually denoted by \( K \), while the time is called the maturity and is usually denoted by \( T \). Thus, if the company enters a foreign exchange (FX) call option for 1,000,000 USD with \( K = 7 \) SEK/USD and \( T = 1 \) year, then it will be able to buy the supplies for no more than 7,000,000 SEK. In addition, if the market rate drops, then the company can buy the dollars at the market rate, since there is no obligation to use the option.

The question remains what the company has to pay for this contract. Or rather, what is the fair price of such a contract today? At maturity, the question is easier to answer. If the rate is below the strike then the contract is worthless, since it makes no sense to buy the asset at a price above the market price. If the market price is above the strike, then the amount of money saved by the company is the difference between the two. Thus, the payoff at maturity for the holder of the contract is

\[
\Phi(X_T) = N \max\{X_T - K, 0\},
\]

where \( K = 7 \), \( N = 1,000,000 \), and where \( X_T \) is the USD/SEK exchange rate at maturity. If the price at maturity would be anything other than \( \Phi(X_T) \), then that would make for an arbitrage opportunity. It follows that the fair price at maturity, by no-arbitrage arguments, is exactly equal to \( \Phi(X_T) \).

But then what is the fair price today? In addition, how can the seller of the option be sure to manage the risk, so that they do not lose money on the trade? Well as it turns out, the answer to both questions coincide. The theory of arbitrage-free pricing provides, given a model for the underlying asset, a fair price for financial derivatives. In addition, the price coincides with the cost of hedging the contract, i.e. for eliminating the risks, and also tells us how to construct the hedge.

**An overview of arbitrage-free pricing theory**

Assume that there are only two assets in the market, a bank account and a stock. We define a bank account process \( B = (B_t)_{t \geq 0} \), where \( B_t \) is the value of one unit of money invested in a bank account at time 0, which has been accumulated with a risk-free interest rate up to time \( t \). The stock process is denoted by \( S = (S_t)_{t \geq 0} \), where \( S_t \) is the unit price of the stock at time \( t \).

On this market, we define a portfolio strategy by two variables stating how much of each asset we own at any time \( t \), and we denote this strategy
by $h = (h_t)_{t \geq 0}$, where $h_t = (x_t, y_t)$ for $t \geq 0$. Here, $x_t$ is the units of $B_t$ we hold at time $t$, and $y_t$ is the number of $S_t$ we hold at time $t$. Then the value of our portfolio at time $t$, denoted $V^h_t$, is

$$V^h_t = x_t B_t + y_t S_t, \quad t \geq 0.$$  

We assume that we are allowed to enter into short positions in the assets, i.e. we may sell the assets, and that we can own any fractional number of assets. Thus, $h_t$ can take values in the whole of $\mathbb{R}^2$.

Intuitively, an arbitrage portfolio is a risk-less portfolio which has a chance of turning a profit. Mathematically, this translates into the following definition of an arbitrage:

$$V^h_0 = 0,$$
$$P(V^h_T > 0) > 0,$$
$$P(V^h_T \geq 0) = 1,$$

for some $T > 0$ and some portfolio strategy $h$. That is, the cost of following the strategy $h$ is zero initially, the final value of it has a non-zero probability of being strictly positive, and the probability that the final value is negative is zero.

This definition assumes that the portfolio strategy $h$ is self-financing, which means that it does not include any injection or withdrawal of capital. Hence, all the changes in the value of the portfolio comes from changes in the value of the underlying assets. In the market we defined earlier, this means that

$$dV^h_t = x_t dB_t + y_t dS_t.$$  

The call option is an example of a contingent claim, i.e. a future stochastic claim which is contingent on the value of the underlying asset. Assume that there exists a contingent claim $\Phi(S_T)$, such that for a particular portfolio strategy $h$, we have that

$$V^h_T = \Phi(S_T)$$  

almost surely. If the cost of the contingent claim at any time $t < T$ would differ from $V^h_t$, then there would be an arbitrage opportunity in the market. One could simply buy the claim and sell the portfolio, or the other way around, to make money without taking any risks. Thus, if such a portfolio exists then it tells us the value of the claim for any $t \leq T$. In addition, the seller of the contingent claim knows exactly how to hedge the claim: by
following the strategy $\tilde{h}$. The portfolio $\tilde{h}$ is called a replicating portfolio, and the existence depends on the market we are studying. If there is a replicating portfolio for every contingent claim, then the market is said to be complete.

There are two fundamental theorems in the arbitrage-free pricing theory which relates the notions of no arbitrage and complete markets to the existence of so called martingale measures. Such a measure is characterized by that every asset process in the market, denominated with the numéraire of the measure, is a martingale under the measure. Martingale measures are sometimes also referred to as risk-neutral measures, reflecting that an arbitrage-free pricing formula is in a sense the value of the claim to a risk-neutral investor.

When the numéraire of the martingale measure is the bank account process, then the measure is usually denoted by $Q$. Thus, under $Q$ we have that

$$X_t = B_t E^Q \left[ \frac{X_T}{B_T} \right], \quad 0 \leq t \leq T,$$

for each asset $X$. The measure $Q$ is particularly suitable for theoretical work, since the idea of a bank account process as a numéraire is intuitive. We simply value all assets in terms of money in the bank. However, defining the bank account process involves defining and modeling a short-rate, which is not directly observable in the market.

If we assume that, under the measure $Q$, the assets follow the dynamics

$$dB_t = rB_t dt,$$
$$dS_t = rS_t dt + \sigma S_t dW_t, \quad t \geq 0,$$

where $r$ and $\sigma$ are constants and where $W = (W_t)_{t \geq 0}$ is a standard Brownian motion, then we have the setting of the famous Black-Scholes formula. With this definition, the market is complete.

There is another measure which can be more suitable for practical work. The forward measure is defined by using the price of a zero-coupon bond as numéraire. A zero-coupon bond pays one unit money at maturity, and pays no interest or coupons until then. The price at time $t$ of a zero-coupon bond, which has maturity $T$, is usually denoted by $P(t, T)$. Immediately, we have that

$$P(t, T) < 1, \quad t < T$$
$$P(T, T) = 1.$$

Specifying a forward measure thus means that we specify a maturity $T$, and the corresponding measure is usually denoted by $Q^T$. For expectations
though, it is common to use shorthand for the measure and simply use $T$ as a superscript. Hence, we have that

$$X_t = P(t, T)\mathbb{E}^T\left[\frac{X_T}{P(T, T)}\right] = P(t, T)\mathbb{E}^T[X_T], \quad 0 \leq t \leq T,$$

for every asset $X$ in the market. An advantage of this measure is that the bond prices $P(t, T)$ are, though perhaps through some interpolation, observable in the market.

The first fundamental theorem of arbitrage-free pricing states that the existence of a martingale measure is equivalent to that there is no arbitrage in the market, while the second fundamental theorem states that the uniqueness of this measure is equivalent to the market being complete. So by these theorems, we can be assured that if the model is well formulated then we can write the fair price, and thus the initial cost of the replicating portfolio, of any claim as

$$V_t = P(t, T)\mathbb{E}^T[\Phi(S_T)], \quad 0 \leq t \leq T.$$

The payoff function (i.e. $\Phi$) does not have to be a function of the asset price at a single point in time; the results can be shown to hold even if the payoff is a function of entire trajectories as well. For this text though, we only consider payoff functions of the form of the example. For further reference on arbitrage-free pricing theory, the reader is referred to [1] and the references therein.

Computing an expectation to price a derivative means employing a numerical method, except in some special cases. The available numerical methods can in general be divided into two approaches, the discretization approach and the simulation approach.

There is a link between the expectation of stochastic processes and PDEs which is established by the Feynman-Kac formula. More precisely, if a conditional expectation is considered to be a function of time and space, then that function can be shown to solve a deterministic PDE that is characterized by the stochastic process. Using this link, it can be shown that the price of a financial derivative, which is a conditional expectation, can be computed by solving a PDE.

The link established by the Feynman-Kac formula is the foundation for the discretization approach to pricing financial derivatives. In our example, the PDE is the Black-Scholes formula. Equivalently, one can discretize the model into a pricing tree, where the type of tree corresponds to the finite difference scheme used to solve the PDE numerically. The discretization approach provide efficient algorithms for computing the prices, given that
the number of dimensions in the model is not too high. This follows since every discretization must use a grid, and the number of grid points increases very fast with the dimension. This problem is commonly referred to as the curse of dimensionality.

If there are too many dimensions in the model, we have to resort to Monte Carlo simulation. This is a very intuitive approach for someone who has a background in probability or statistics, since it means using the law of large numbers. That is, a large number of paths of the underlying is simulated, the payoff is computed for each path, and the average is used to estimate the expectation.

The European call option is just one example of a financial derivative, and there seems to be no end to the variations on derivatives trading around the world. Examples include forwards, futures, swaps, Asian types, barrier types etc. A specific type which we will consider is the Bermudan callable. Such a contract states that the owner has the right, but not obligation, to exercise at a discrete set of times before maturity. So if the option we considered earlier was Bermudan with exercise times \( t_1 < t_2 < \ldots < t_n = T \), then at any time \( t_i, i = 1, \ldots, n \), the owner can choose to exercise and receive the payoff \( \Phi(S_{t_i}) \).

Assuming a non-negative claim, a rational investor will only exercise if \( \Phi(S_{t_i}) > 0 \). If that holds, when is it optimal to exercise and how do we price such claims? Consider a stopping strategy defined by the stopping time \( \tau \). The fair value given that stopping time, by no-arbitrage arguments, is

\[
V_t = P(t, T) \mathbb{E}^T[\Phi(S_\tau)|\mathcal{F}_t], \quad 0 \leq t \leq T.
\]

Assume now that there is another stopping strategy \( \tilde{\tau} \) such that

\[
\mathbb{E}^T[\Phi(S_\tau)|\mathcal{F}_t] < \mathbb{E}^T[\Phi(S_{\tilde{\tau}})|\mathcal{F}_t], \quad 0 \leq t \leq T.
\]

Then by no-arbitrage arguments again, the fair value must be

\[
V_t = P(t, T) \mathbb{E}^T[\Phi(S_\tau)|\mathcal{F}_t], \quad 0 \leq t \leq T.
\]

Repeating this argument, we come to the conclusion that the fair value of a Bermudan callable is found by exercising the claim optimally. That is,

\[
V_t = \mathop{\text{ess sup}}_{\tau \in \mathcal{T}_t} P(t, T) \mathbb{E}^T[\Phi(S_\tau)|\mathcal{F}_t], \quad 0 \leq t \leq T,
\]

where \( \mathcal{F} = (\mathcal{F}_t)_{t \geq 0} \) is the filtration of the model and where \( \mathcal{T}_t \) is the set of admissible stopping times \( \tau \), such that \( t \leq \tau \leq T \). In our example, admissible
means that the stopping times can only take values in a subset of \( \{ t_i \}_{i=1}^n \). Thus, the price of Bermudan contracts admits the representation of a Snell envelope. This is a central theme in this thesis and so we give it a little extra attention in the next subsection.

Pricing a Bermudan derivative with a discretization approach is straightforward. For each time when the derivative can be exercised, the expected payoff when not exercising, i.e. the continuation value, is given directly by the solution to the PDE. Thus, we can simply compare the exercise value with the continuation value for each point in the grid, and set the value of the contract in each point to the maximum of the two. In the Monte Carlo approach the continuation value is not given as easily, so estimating the optimal stopping strategy can be more difficult. The last two papers in this thesis are focused on handling the Bermudan feature for Monte Carlo approaches.

The Snell envelope and its connection to RBSDEs

The purpose of the description which follows here is to give a brief introduction to some important concepts in the thesis. Hence, some regularity conditions and space definitions will be omitted.

Let \( U = (U_t)_{0 \leq t \leq T} \) be an \( \mathbb{R} \)-valued càdlàg (right continuous with left limits) process which is adapted to a filtration \( \mathcal{F} \). Then there exists an \( \mathcal{F} \)-adapted \( \mathbb{R} \)-valued càdlàg process \( Z = (Z_t)_{0 \leq t \leq T} \) such that \( Z \) is the smallest supermartingale which dominates \( U \). That is, if \( (\bar{Z}_t)_{0 \leq t \leq T} \) is another càdlàg \( \mathbb{R} \)-valued \( \mathcal{F} \)-adapted supermartingale such that \( \bar{Z}_t \geq U_t \) almost surely for \( 0 \leq t \leq T \), then \( \bar{Z}_t \geq Z_t \) for \( 0 \leq t \leq T \).

The process \( Z \) is called the Snell envelope of \( U \) and it has the following properties.

- For any \( \mathcal{F} \)-stopping time \( \theta \) it holds that
  \[
  Z_\theta = \text{ess sup}_{\tau \in \mathcal{T}_\theta} \mathbb{E}[U_\tau | \mathcal{F}_\theta] \quad (\text{and hence } Z_T = U_T).
  \]

- The Doob-Meyer decomposition of \( Z \) implies the existence of a continuous martingale \( (M_t)_{0 \leq t \leq T} \) and two nondecreasing predictable processes \( (A_t)_{0 \leq t \leq T} \) and \( (B_t)_{0 \leq t \leq T} \) which are, respectively, continuous and purely discontinuous, such that for all \( 0 \leq t \leq T \) we have
  \[
  Z_t = M_t - A_t - B_t \quad (\text{with } A_0 = B_0 = 0).
  \]
• For any $0 \leq t \leq T$,
\[
\{ \Delta B_t > 0 \} \subseteq \{ \Delta U_t < 0 \} \cap \{ Z_{t-} = U_{t-} \}.
\]
Hence, if $U$ only has positive jumps, then $Z$ is a continuous process.

• If $\theta$ is an $\mathcal{F}$-stopping time then
\[
\tau^*_\theta := \inf \{ t \geq \theta : Z_t = U_s \} \wedge T
\]
is optimal after $\theta$, i.e.
\[
Z_0 = \text{ess sup}_{\tau \in T_0} \mathbb{E}[U_\tau | \mathcal{F}_\theta] = \mathbb{E}[Z_{\tau^*_\theta} | \mathcal{F}_\theta] = \mathbb{E}[Z_{\tau^*_\theta} | \mathcal{F}_0].
\]

The Snell envelope is strongly connected to RBSDEs. Given some in-data $(\xi, f, O)$, where $\xi$ is a random variable, $f$ is a function and $O = (O_t)_{0 \leq t \leq T}$ is a càdlàg process, the solution to the RBSDE associated with the in-data is a triplet of processes $(Y, Z, K)$ such that
\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + (K_T - K_t) - \int_t^T Z_sdB_s, \quad 0 \leq t \leq T,
\]
\[
Y_t \geq O_t, \quad 0 \leq t \leq T,
\]
\[
\int_0^T (Y_t - O_t)dK_t = 0. \tag{1}
\]
The processes $(Y, Z, K)$ are all defined in appropriate spaces which are made precise in papers I and II, but in particular, the process $K$ is a non-decreasing process starting at $K_0 = 0$. This process is what “pushes” the solution upwards to keep it above $O$, and thus makes the second statement hold true for all $t$. Furthermore, the last condition tells us that the “pushing” is minimal in the sense that it only occurs when $Y$ hits the obstacle $O$.

The connection between the solution of a RBSDE and the Snell envelope is shown by the following result from [7], in which we omit the regularity conditions. Let $(Y, K, Z)$ be a solution to (1). Then for each $0 \leq t \leq T$ it holds that
\[
Y_t = \text{ess sup}_{\tau \in T_t} \mathbb{E}\left[ \int_t^\tau f(s, Y_s, Z_s)ds + S_{\tau}1_{\{\tau<T\}} + \xi 1_{\{\tau=T\}} | \mathcal{F}_t \right],
\]
where $T_t$ is the set of all stopping times $\tau$ such that $t \leq \tau \leq T$, and the converse also holds.

This connection is a very important ingredient in the first two papers of this thesis, which concerns optimal switching. A brief introduction to that problem setting will be given below.
An overview of optimal switching

Arbitrage-free pricing theory has been a useful tool in the literature on what is known as real options, where one is concerned with choices faced by businesses. Examples of such decisions include the option to defer, abandon or alter projects, to expand or contract business lines, or to switch inputs or outputs for the company’s production.

Some authors model these choices as a series of, possibly entangled, call or put options (a put option gives the right of selling the asset instead buying it), and use no-arbitrage arguments to derive the value of a firm or a project. The difference is that the underlying is now e.g. expected cash-flows or yields resulting from the decisions made.

As an example, consider an investment decision where the size of the investment is $K$, and the expected future earnings from this investment is $V$. Rational companies will only make the investment if the expected earnings is larger than the initial cost, so the expected payoff of the investment is

$$\max\{V - K, 0\},$$

at the time when the company can do the investment. Assume that the company can choose to either invest today or postpone the investment to another time, say $t_1$. Also, at time $t_1$ the same choice is present until the next time $t_2$, and so on. If there are $n$ investment opportunities, then the decision is equivalent to a Bermudan call option with exercise times $t_1 < t_2 < \ldots < t_n$. Thus, the value of the decision is equivalent to the price of a Bermudan call option.

Some early examples of papers where real investments are analyzed with the same type of tools present in arbitrage-free pricing theory include [2] where the authors use no-arbitrage arguments to derive the value of a resource producing mine as a function of the price of the resource and the production decisions made, and [3] where the author investigates a firm’s optimal entry and exit decision when the price of the produced good is a random walk. In the latter, idle firms and active firms are viewed as assets which are call options to each other. For further reference of real options the reader is referred to [13] and [12].

The entry and exit decision problem in [3] is an example of a switching decision, which is a particular type of real option. This type of optimal switching problem has been of much interest in the literature in the past years. The optimal switching problem in general refers to the decision to switch between a set of modes to maximize some functional of the decisions, where the functional also can depend on some driving stochastic processes.
As an example, consider a company that produces electricity. The electricity cannot be stored and has to be sold directly, at a price which is stochastic. Hence, the production of the electricity is the only variable the company can alter to affect the profit. If the company only can choose to let the production be active or inactive, then this is an example of a two-modes optimal switching problem, where the driving process is the price of electricity. An example of a multiple-modes switching problem is when a company has the option to choose between different product lines, each with a different stochastic production cost and profit.

Now consider an optimal switching problem where we also have to decide when to switch. Given the entangled call option view of [3] and the Snell envelope representation of a price, it is not far fetched to believe that an optimal switching strategy, and the corresponding value of decision, amounts to a system of entangled Snell envelopes.

Let the problem be of two-modes, 1 and 2, and let the profit per unit time in each state be \( \psi_i(t) \). Furthermore, assume that the time frame is finite and that at maturity, \( T \), the final profit from the active state is a random variable, \( \xi_1 \) if state 1 is active and \( \xi_2 \) if state 2 is active. If there are no costs incurred when switching between the two states, then the system becomes

\[
Y_t^1 = \text{ess sup}_{\tau \geq t} \mathbb{E} \left[ \int_t^\tau \psi_1(s) ds + Y_{\tau}^2 1_{\{\tau < T\}} + \xi_1 1_{\{\tau = T\}} \middle| \mathcal{F}_t \right],
\]

\[
Y_t^2 = \text{ess sup}_{\tau \geq t} \mathbb{E} \left[ \int_t^\tau \psi_2(s) ds + Y_{\tau}^1 1_{\{\tau < T\}} + \xi_2 1_{\{\tau = T\}} \middle| \mathcal{F}_t \right],
\]

where \( \mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T} \) is the filtration of the model, and where \( Y_t^i \) is the optimal expected profit given that state \( i \) is active at time \( t \).

Provided the existence of continuous processes which satisfy this system, the optimal switching strategy is fully characterized by the theory of Snell envelopes. Proving the existence can be done by using the connection between the Snell envelope and RBSDEs, and this is the technique we use in the main proofs of the first two papers in this thesis.
Summary of the papers

Paper I: A Full Balance Sheet Two-modes Optimal Switching problem

In this paper we consider an optimal switching problem in two-modes, where we take into account both profit and cost yields in each mode $i$, denoted $\psi^+_i$ and $\psi^-_i$. We allow for two possible actions, switching and termination, and the chosen action is based on the optimal expected future profit and cost yields in each mode.

If the optimal expected profit is too low as compared to the optimal expected cost yield in one of the modes, then the mode is terminated. If the optimal expected profit is too low as compared to the optimal expected profit of the other mode, then we switch the active mode. Furthermore, the model allows for switching based on both sides of the balance sheet, so that e.g. the optimal expected costs of the two modes are compared. An application is a company with two projects which only has the option to switch active project or to terminate.

The optimal switching problem is formulated with the following system of Snell envelopes

\[
Y_t^{+\cdot}_i = \operatorname{ess} \sup_{\tau \geq t} \mathbb{E} \left[ \int_t^\tau \psi^+_i(s, \omega, Y^+_s, Z^+_s) ds + S^+_\tau 1_{\{\tau < T\}} + \xi^+_i 1_{\{\tau = T\}} \bigg| \mathcal{F}_t \right],
\]

\[
Y_t^{-\cdot}_i = \operatorname{ess} \inf_{\tau \geq t} \mathbb{E} \left[ \int_t^\tau \psi^-_i(s, \omega, Y^-_s, Z^-_s) ds + S^-_\tau 1_{\{\tau < T\}} + \xi^-_i 1_{\{\tau = T\}} \bigg| \mathcal{F}_t \right],
\]

for $i = 1, 2$, where the obstacles are defined as

\[
S^+_{t,1} = (Y^+_{t,2} - \ell_1(t)) \lor (Y^-_{t,1} - a_1(t)),
\]

\[
S^+_{t,2} = (Y^+_{t,1} - \ell_2(t)) \lor (Y^-_{t,2} - a_2(t)),
\]

\[
S^-_{t,1} = (Y^-_{t,2} + \ell_1(t)) \land (Y^+_{t,1} + b_1(t)),
\]

\[
S^-_{t,2} = (Y^-_{t,1} + \ell_2(t)) \land (Y^+_{t,2} + b_2(t)),
\]

and where $\ell_i, a_i$ and $b_i$ are switching costs.

Using an increasing sequence of approximating systems of RBSDEs, we
prove the existence of a minimal solution to the following system of RBSDEs

\[ Y_t^{+i} = \xi_t^+ + \int_t^T \psi_t^+(s, \omega, Y_s^{+i}, Z_s^{+i}) ds + (K_T^{+i} - K_t^{+i}) - \int_t^T Z_s^{+i} dB_s, \]

\[ Y_t^{-i} = \xi_t^- + \int_t^T \psi_t^-(s, \omega, Y_s^{-i}, Z_s^{-i}) ds - (K_t^{-i} - K_t^{+i}) - \int_t^T Z_s^{-i} dB_s, \]

\[ Y_t^{+i} \geq S_t^{+i}, \quad Y_t^{-i} \leq S_t^{-i}, \]

\[ \int_0^T (Y_t^{+i} - S_t^{+i}) dK_t^{+i} = 0, \quad \int_0^T (S_t^{-i} - Y_t^{-i}) dK_t^{-i} = 0, \]

which is equivalent to the system (2). In addition, since the optimal stopping time of the Snell envelopes is known, the optimal switching strategy is fully characterized. A counter-example shows that uniqueness does not hold.

**Paper II: A Two-modes Mean-field Optimal Switching Problem for The Full Balance Sheet**

The paper considers an optimal switching setting with many nodes, where each node has two modes. An example where this is a relevant formulation is a company with many production lines, where each line can be set to high-production mode or low-production mode. The model allows for both profit and cost yields and we assume that the profit and cost yields per unit time of is coupled through the average of all yields. That is, we compare the production lines to the average as a benchmark.

Using the notation of the summary of Paper I, except that we add the index \(j = 1, \ldots, N\) to denote production line \(j\), we have

\[ \psi_t^{+j}(t) = \psi_t^+(t, Y_t^{+j}, \frac{1}{N} \sum_{j=1}^N Y_t^{+j}), \]

\[ \psi_t^{-j}(t) = \psi_t^-(t, Y_t^{-j}, \frac{1}{N} \sum_{j=1}^N Y_t^{-j}). \]

Since the production lines are interconnected through the coupling term, the corresponding system of Snell envelopes is highly complex. To handle this, we instead consider the aggregate expected yields of all projects, and use a mean-field approximation \(EY_t^{\pm,1}\) for the coupling term. The corresponding
system of Snell envelopes becomes:

$$ Y_{t}^{+,i} = \text{ess sup}_{\tau \geq t} \mathbb{E} \left[ \int_{t}^{\tau} \psi_{t}^{+}(s, Y_{s}^{+,i}, \mathbb{E}Y_{s}^{+,i})ds + S_{\tau}^{+,i}1_{[\tau < T]} + \xi_{\tau}^{+}1_{[\tau = T]} \right] | \mathcal{F}_{t}, $$

$$ Y_{t}^{-,i} = \text{ess inf}_{\tau \geq t} \mathbb{E} \left[ \int_{t}^{\tau} \psi_{t}^{-}(s, Y_{s}^{-,i}, \mathbb{E}Y_{s}^{-,i})ds + S_{\tau}^{-,i}1_{[\tau < T]} + \xi_{\tau}^{-}1_{[\tau = T]} \right] | \mathcal{F}_{t}, $$

for $i = 1, 2$, where

$$ S_{t}^{+,1} = (Y_{t}^{+,2} - \ell_{1}(t)) \lor (Y_{t}^{-,1} - a_{1}(t)), $$

$$ S_{t}^{+,2} = (Y_{t}^{+,1} - \ell_{2}(t)) \lor (Y_{t}^{-,2} - a_{2}(t)), $$

$$ S_{t}^{-,1} = (Y_{t}^{-,2} + \ell_{1}(t)) \land (Y_{t}^{+,1} + b_{1}(t)), $$

$$ S_{t}^{-,2} = (Y_{t}^{-,1} + \ell_{2}(t)) \land (Y_{t}^{+,2} + b_{2}(t)). $$

In terms of BSDEs, the system is equivalent to the following system of MF-RBSDEs:

$$ Y_{t}^{+,i} = \xi_{i}^{+} + \int_{t}^{T} \psi_{t}^{+}(s, Y_{s}^{+,i}, \mathbb{E}Y_{s}^{+,i})ds + (K_{T}^{+,i} - K_{t}^{+,i}) - \int_{t}^{T} Z_{s}^{+,i}dB_{s}, $$

$$ Y_{t}^{-,i} = \xi_{i}^{-} + \int_{t}^{T} \psi_{t}^{-}(s, Y_{s}^{-,i}, \mathbb{E}Y_{s}^{-,i})ds - (K_{T}^{-,i} - K_{t}^{-,i}) - \int_{t}^{T} Z_{s}^{-,i}dB_{s}, $$

$$ Y_{t}^{+,i} \geq S_{t}^{+,i}, \quad Y_{t}^{-,i} \leq S_{t}^{-,i}, $$

$$ \int_{0}^{T} (Y_{t}^{+,i} - S_{t}^{+,i})dK_{t}^{+,i} = 0, \quad \int_{0}^{T} (S_{t}^{-,i} - Y_{t}^{-,i})dK_{t}^{-,i} = 0. $$

In this paper we consider a slightly more general system of MF-RBSDEs:

$$ Y_{t}^{+,i} = \xi_{i}^{+} + \int_{t}^{T} \psi_{t}^{+}(s, Y_{s}^{+,i}, \mathbb{E}Y_{s}^{+,i}, Z_{s}^{+,i})ds + (K_{T}^{+,i} - K_{t}^{+,i}) - \int_{t}^{T} Z_{s}^{+,i}dB_{s}, $$

$$ Y_{t}^{-,i} = \xi_{i}^{-} + \int_{t}^{T} \psi_{t}^{-}(s, Y_{s}^{-,i}, \mathbb{E}Y_{s}^{-,i}, Z_{s}^{-,i})ds - (K_{T}^{-,i} - K_{t}^{-,i}) - \int_{t}^{T} Z_{s}^{-,i}dB_{s}, $$

$$ Y_{t}^{+,i} \geq S_{t}^{+,i}, \quad Y_{t}^{-,i} \leq S_{t}^{-,i}, $$

$$ \int_{0}^{T} (Y_{t}^{+,i} - S_{t}^{+,i})dK_{t}^{+,i} = 0, \quad \int_{0}^{T} (S_{t}^{-,i} - Y_{t}^{-,i})dK_{t}^{-,i} = 0. $$

This setup is similar to the model we used in Paper I, but with an important difference: the yields are allowed to depend on the mean of the expected yields. Because of the added generality we had to prove some additional results regarding MF-RBSDEs. We provide existence and uniqueness of
the MF-RBSDEs considered, a comparison result, and a uniform bound on the solution of the MF-RBSDE given the in-data. With these prerequisite results, we were able to use a procedure similar to the one in Paper I to prove existence of a minimal solution to the system [3].

**Paper III: Pricing Bermudan options - A non-parametric estimation approach**

We propose a new method for estimating continuation values used to find approximations of optimal stopping strategies for pricing Bermudan options with a Monte Carlo approach. Let \( \{t_i\}_{i=1}^n \) be the times where we are allowed to exercise the Bermudan option to receive the payoff \( Z_t = \Phi(S_t) \), where \( S_t \) is the value of the underlying asset at time \( t \) and where

\[
0 < t_1 < \ldots < t_n = T.
\]

In addition, let \( \hat{\tau}_i \) denote the optimal stopping time after \( t_i \), so that the value of the contract at each time \( t_i \) can be written as

\[
V_{t_i} = \text{ess sup}_{\tau \in \{t_k\}_{k=1}^n} \mathbb{E}[Z_\tau | S_{t_i}] = \mathbb{E}[Z_{\hat{\tau}_i} | S_{t_i}],
\]

since we are assuming a Markovian setting for the model. Here, we are w.l.o.g. considering \( V \) and \( Z \) to be discounted with the numéraire of the martingale measure, and omitting the notation of the measure.

Now assume that we for each time \( t_i \) know the function \( f_i \), where

\[
f_i(S_{t_i}) = \mathbb{E}[\Phi(S_{\hat{\tau}_i+1}) | S_{t_i}],
\]

i.e. we know the continuation value at each \( t_i \). Then, the optimal strategy is given directly by dynamic programming. Indeed, we have that

\[
\hat{\tau}_n = T,
\]

\[
\hat{\tau}_i = \begin{cases} 
  t_i, & \text{if } \Phi(S_{t_i}) \geq f_i(S_{t_i}), \\
  \hat{\tau}_{i+1}, & \text{otherwise}.
\end{cases}
\]

That is, we can recursively find the optimal stopping time for each simulation in our Monte Carlo approach, and then average the stopped payoffs to find an estimate for the price.

Assuming that \( \hat{\tau}_{i+1} \) is known, how do we find \( f_i \)? We know that for each \( i \) it holds that

\[
f_i = \arg\min_{g \in L^2} \mathbb{E}[(\Phi(S_{\hat{\tau}_{i+1}}) - g(S_{t_i}))^2].
\]
If we parameterize the function estimator, we can use least-squares regression to find the best estimate of the form of the parameterization. This has been done in [11], where the authors use basis functions to parameterize the function estimator \( \tilde{f}_i \).

If we want a non-parametric estimate, or if we use an overly flexible parameterization, then the least-squares regression will be ill-posed due to overfitting. The aim of this paper is to handle the overfitting by using a regularization part in the formulation of the estimate \( \tilde{f}_i \), so that we penalize the formulation for solutions which are not smooth. For details on regularization methods in general the reader is referred to [8].

Using Itô’s formula we can derive a PDE which characterizes the underlying process, to which the solution is \( f_i \). That is, we can derive a generator \( \mathcal{A} \) such that

\[
(\partial_t + \mathcal{A})f_i(x) = 0, \quad x \in \mathcal{S},
\]

where \( \mathcal{S} \) is the state space of the model. It follows that

\[
\min_{g \in L^2} \mathbb{E}\left[ ((\partial_t + \mathcal{A})g(S_{t_i}))^2 \right] = \mathbb{E}\left[ ((\partial_t + \mathcal{A})f_i(S_{t_i}))^2 \right] = 0.
\]

Since we know that the real solution \( f_i \) has zero norm, we should expect a good estimate to have a reasonably small norm. Thus, we can use a discretization of the PDE evaluated in the function estimator as a regularization term, which is the main idea of this paper. We should note that even though minimizing the norm of the PDE might be more difficult than to solve the PDE, it is less costly to compute the norm of a guess of the solution than to solve the PDE. This makes makes it viable for a regularization part in the formulation.

We combine the two problems above and consider a discretization of the following

\[
\min_{g \in L^2} \mathbb{E}\left[ ((\partial_t + \mathcal{A})g(S_{t_i}))^2 \right] + \lambda \mathbb{E}\left[ ((\partial_t + \mathcal{A})f_i(S_{t_i}))^2 \right],
\]

for some regularization parameter \( \lambda > 0 \). The particular discretization considered in this paper is described in more detail below.

As an estimator of \( f_i \), we use a simple function on a grid:

\[
\tilde{f}_i(x) = \sum_{k=1}^{N} \tilde{y}_k 1_{\{x \in A_k\}},
\]

where

\[
A_k = \{ x \in \mathcal{S} : \| x - z_k \|_2 \leq \| x - z_p \|_2, \quad p = 1, \ldots, N \}, \quad k = 1, \ldots, N,
\]
is the Voronoi tessellation of $S$, defined by the grid points $\{z_k\}_{k=1}^N$.

The constants $\{\tilde{y}_k\}_{k=1}^N$ are determined by solving the following regularized least-squares problem, which is the discrete version of (4):

$$
\min_{\{y_k\}_{k=1}^N} \sum_{j=1}^m \left( \Phi(S^{(j)}_{t_{i+1}}) - \tilde{y}_{k_j} \right)^2 + \lambda \sum_{k=2}^{N-1} \left( \frac{V^{i+1,k}_{t_{i+1}} - \tilde{y}_k}{t_{i+1} - t_i} + a_k y_{k-1} + b_k y_k + c_k y_{k+1} \right)^2,
$$

where $V^{i+1,k}_{t}$ is the computed value of the contract in the grid point $z_k$ at time $t_{i+1}$. Here, $j = 1, \ldots, m$ is the index of the simulation outcomes, $k_j = \{ \ell \in \{1, \ldots, N\} : S_{t_{i+1}}^{(j)} \in A_\ell \}$

is the index of the tessellation set which $S_{t_{i}}^{(j)}$ landed in, and where $a_k$, $b_k$, and $c_k$ are constants determined by the finite difference scheme employed to discretize the operator $A$ in the grid points. The parameter $\lambda$ states how much emphasis we wish to put on the smoothness of the solution.

It is proven that this function estimator converges to $f_t$ when the number of simulations and grid points tend to infinity, given some assumptions on the convergence of the grid. This result is actually shown to hold for all finite regularization functions, and not just when we regularize with the PDE. Finally, we show that the method works well in one dimension with some numerical examples.

Paper IV: PDE-regularization for pricing multi-dimensional Bermudan options with Monte Carlo simulation

The method of Paper III is extended to the multi-dimensional setting by relaxing the use of a grid, and by adding local weights for stability. Removing the grid is important since any grid means that we are still affected by the curse of dimensionality, which limits the use of the method.

Using the notation of the summary of Paper III, we set the function estimator $\tilde{f}_t$ to be a simple function on a Voronoi tesselation of $S$, but this time defined by the simulation outcomes. That is, we set

$$
\tilde{f}_t(x) = \sum_{j=1}^m \tilde{y}_j \mathbf{1}_{\{x \in A_j\}},
$$

where

$$
A_j = \left\{ x \in S : \|x - S_{t_{i}}^{(j)}\|_2 \leq \|x - S_{t_{i}}^{(p)}\|_2 , \quad p = 1, \ldots, m \right\}, \quad j = 1, \ldots, m,
$$
is the Voronoi tessellation of $S$ defined by the simulation outcomes \( \{ S_{t_i}^{(j)} \}_{j=1}^m \).

The corresponding discrete version of \( \text{(4)} \) is the following:

\[
\min \sum_{j=1}^m \left( \Phi(S_{t_i}^{(j)}) - y_j \right)^2 + \lambda \sum_{j=2}^{m-1} \left( \frac{V_{t_i+1}^{(j)} - y_j}{t_{i+1} - t_i} + a_j y_{j-1} + b_j y_j + c_j y_{j+1} \right)^2,
\]

where \( V_{t_i+1}^{(j)} \) is the computed value of the contract in the point \((t_{i+1}, S_{t_i}^{(j)})\), and where \( a_j, b_j \) and \( c_j \) are determined by the finite difference scheme employed to discretize the operator $\mathcal{A}$ in the simulation outcomes \( \{ S_{t_i}^{(j)} \}_{j=1}^m \).

This can can be written as an equation system

\[
Ay = b,
\]

where the matrix $A$ is symmetric and pentadiagonal. Thus, solutions to the system \( \text{(5)} \) can be obtained quite efficiently.

However, this assumes that we have an ordering of the simulation outcomes. In one dimension the ordering is naturally the sorted order, i.e.

\[
S_{t_i}^{(1)} \leq S_{t_i}^{(2)} \leq \ldots \leq S_{t_i}^{(m)} ,
\]

but in higher dimensions it is not as clear how to define a good ordering. Noting that finite differences are better approximations for small distances, it seems plausible that a good ordering minimizes the total distance between the points. That is, we seek a permutation of the simulation indices, \( \{ k_j \}_{j=1}^m \), such that

\[
\{ k_j \}_{j=1}^m = \arg\min_{p_1, \ldots, p_m} \sum_{j=1}^{m-1} \| S_{t_i}^{(p_{j+1})} - S_{t_i}^{(p_j)} \|_2 .
\]

Hence, with this criterion for finding an ordering we have to solve the traveling salesman problem (TSP problem), which is NP-hard. Fortunately, there are many methods of finding approximate solutions. In this paper, we use a genetic algorithm for finding an approximate solution to the TSP problem.

Another thing which we have to address is the numerics of solving the system \( \text{(5)} \). If the distribution of the underlying is non-uniform, then the distance between the simulation outcomes in the tail of the distribution can be very large as compared to the rest of the simulation outcomes. Now since the \( a, b, c \) stems from the use of finite differences, they will be proportional to the inverse of the distances. Consequently, the elements of the matrix in \( \text{(5)} \) can become very different in magnitude, which results in ill-conditioning.
To handle this we add local weights in the formulation. That is, we introduce weights \( \{ \gamma_j \}_{j=1}^m \), and then solve the following problem:

\[
\min_{\{y_j\}_{j=1}^m} \sum_{j=1}^m \left( \Phi(S_{j+1}^{(j)} - y_j) \right)^2 + \lambda \sum_{j=2}^{m-1} \gamma_j^2 \left( \frac{V_j^{(j+1)} - y_j}{t_{j+1} - t_i} + a_j y_{j-1} + b_j y_j + c_j y_{j+1} \right)^2.
\]

Setting the local weights proportional to the distances will help to keep the matrix in (5) better-conditioned, and give better results. The intuition is that in areas where there are many simulations, less emphasis is required on the regularization part. By letting the local weights be smaller, we put relatively more weight on the projection part.

Using the multi-dimensional Black-Scholes model and simple payoffs, we illustrate that the method can produce accurate prices of Bermudan derivatives in multi-dimensional settings. A number of open problems regarding the performance are identified, and possible solutions are discussed. As an example, choosing the regularization parameter in a systematic way is an important problem. If these problems can be resolved, then the method is likely to be competitive for pricing multi-dimensional Bermudan options.
References


