Syllogistic Analysis and Cunning of Reason in Mathematics Education

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DISSERTATION

Karlstad University Studies  |  2013:28

ISSN 1403-8099


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Distribution:
Karlstad University
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SE-651 88 Karlstad, Sweden
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Print: Universitetstryckeriet, Karlstad 2013

WWW.KAU.SE
Abstract

This essay explores the issue of organizing mathematics education by means of syllogism. Two aspects turn out to be particularly significant. One is the syllogistic analysis while the other is the cunning of reason. Thus the exploration is directed towards gathering evidence of their existence and showing by examples their usefulness within mathematics education.

The syllogistic analysis and the cunning of reason shed also new light on Chevallard’s theory of didactic transposition. According to the latter, each piece of mathematical knowledge used inside school is a didactic transposition of some other knowledge produced outside school, but the theory itself does not indicate any way of transposing, and this empty space can be filled with the former.

A weak prototype of syllogism considered here is Freudenthal’s change of perspective. Some of the major difficulties in mathematics learning are connected with the inability of performing change of perspective. Consequently, to ease the difficulties becomes a significant issue in mathematics teaching. The syllogistic analysis and the cunning of reason developed in this essay are the contributions to the said issue.
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Introduction

HALFWAY upon the road of our life, I came to myself amid a dark wood where the straight path was confused. And as it is a hard thing to tell of what sort was this wood, savage and rough and strong, which in the thought renews my fear, even so is it bitter; so that death is not much more; but to treat of the good which I there found, I will tell of the other things which there I marked.


The change of perspective, as a didactical principle, is one of the most important contributions made by Freudenthal (1978, pp. 242–252) in the domain of mathematics education. The importance of this principle is very clear as Freudenthal said:

Yet I believe that the observation I reported here is the most important I have been confronted with for quite a while, and I am sure that the questions it gives rise to are the most urgent we are expected to answer. Some of learning difficulties, even for those who are excellent in arithmetic, is caused by being unable to perform change of perspective. (ibid., pp. 245-249)

The situation that the student gazes at a task and says of I do not know where to begin is certainly recognizable for many teachers. The student for some mysterious reason is locked in one perspective or what amounts to the same as trapped in one context. The change of perspective does not mean to overthrow the old perspective, on the contrary, this old perspective is equally valuable as the new one. The change of perspective consists in a synthesis of two different perspectives.

The equation $x + 7 = 11$ was used by Freudenthal in his enunciation on the change of perspective. For my purpose I shall use the equation $2x + 4 = 6$ instead. In general, an equation is a synthesis $A = B$ of two objects $A$ and $B$, as a result, an equation is a syllogism. To solve the equation $2x + 4 = 6$ is to relate it to another equation $x = 1$ through some intermediate equations. I shall exhibit the procedure of relating two equations now, it is

\[
\begin{align*}
2x + 4 &= 6 \\
2x &= 2 \\
x &= 1.
\end{align*}
\]
Altogether there are three equations (syllogisms), and solving equation becomes finding interrelationships among the equations.

Mathematics, according to Encyclopedia Britannica (2012), is the science of structure, order and relation (see the quotation at the beginning of the chapter four). The simplest order is that between two objects, say, \(3 < 5\); and the same holds for the relation. In the sense of the present essay, an order is a syllogism and a relation is a such as well. One of the most important mathematical structures, the group structure, is also a syllogism, for it concerns itself the multiplication of two elements.

Mathematics, according to Encyclopedia Britannica (2012) again, deals with logical reasoning. The logical reasoning in mathematics is either inductive or deductive. The deductive reasoning, before Frege, was completely identified with the Aristotelean syllogistic reasoning. An Aristotelean syllogism has the form of

\[
\begin{align*}
A \rightarrow B \\
B \rightarrow C \\
\Rightarrow \\
A \rightarrow C
\end{align*}
\]

where the sign \(\rightarrow\) means the word is.

The curricula in Sweden after 1962 seem, according to Tomas Englund (oral communication), to evolve in the following fashion:

\[
\text{centralization} \quad \rightarrow \quad \text{de-centralization} \quad \rightarrow \quad \text{re-centralization}
\]

with Lgr 62, Lpo 94 and Lgr 11 as the characteristic documents. The formula is understood in the sense of descriptions on aims and knowledge in schooling. This attitude is in fact confirmed by a recent thesis of Nordin (2012).

Those five paragraphs above exhibit the change of perspective, the equation solving, the group structure, the Aristotelean deductive reasoning, and the development of curricula in Sweden since 1962. A close examination of these examples reveals a common part that they share. They all have something to do with syllogisms and their interrelationships, and these words are in fact the principal words of this essay. As a principle, the exact meanings of the principal concepts used shall not be given here but in the main body of the text.

In a broadest sense, a syllogism is

\[
\text{thesis} \quad \rightarrow \quad \text{antithesis} \quad \rightarrow \quad \text{synthesis}.
\]

In order to be able to talk about interrelationships among syllogisms, we shall treat a syllogism as a structure, just like a water molecule is made up by an oxygen atom and two hydrogen atoms in which case the said interrelationships are on the level of water.

Syllogism and interrelationship being the principal issues, the present essay is constructed with the center at the following questions:
• how can teaching and learning of mathematics be organized by syllogisms?

• what makes the syllogistic organization possible?

The aim here is therefore to inquire into these two fundamental questions and to indicate answers.

Of course, the inquiry starts with a fundamental belief that teaching and learning of mathematics can to a great extent be organized by the syllogism. This belief does not grow from the middle of nowhere, it is an outcome from a huge accumulation of mathematics and mathematics teaching practices. And it should not be taken as an answer to the fundamental questions, not even a partial one, for what concerns mathematics education is not a simple positive answer but more importantly the how-issue in the questions. It is this issue that the essay is principally addressed to.

In the process of the inquiry it is revealed that two aspects of syllogism are particularly significant. The one, which lies outside syllogism, is the syllogistic analysis; and the other lying inside is the cunning of reason. In fact it is these two aspects that are really the underpinnings of the present essay. Thus, teaching and learning of mathematics can be greatly organized by syllogistic analysis and skillful use of the cunning of reason; and the syllogistic organization is possible because of our deductive faculty and the cunning of reason. Now that the importance of these two aspects is clear, the main body of the essay is in the first place to formalize them in such a way that they become suitable instruments in mathematics education. Insofar as instruments are concerned, evidence of their usefulness must be gathered and examples must be showed; and all these occupy the rest of the essay.

The two fundamental questions can be looked upon from four domains: mathematics, mathematics education, philosophy of mathematics, and philosophy of mathematics education; and a choice of course depends on the background of the one who looks at them. Being a researcher and teacher in mathematics for more than twenty-five years, I automatically inquire into the questions from the domain of mathematics (education), especially from the perspective of preparing lectures and that of class instruction.

And at this level the first principle in the how-issue becomes this, it is helpful to carry out a syllogistic analysis on a piece of mathematics to be taught and it is the student who constructs knowledge himself and it is the teacher who, by making a skillful use of the cunning of reason, can supply the student with necessary stimuli needed in the construction.

For instance in the teaching and learning of using wrench, the teacher must experience the change of the wrench as tool to the wrench as material thing, and the student must experience the change of the wrench as material thing to the wrench as tool. These changes are possible because of the cunning of reason.
The cunning of reason, in this essay and as a category, is more mathematical and psychological than philosophical. A skillful use of it depends heavily on a piece of mathematics to be taught and on the background of the student. The dependence can be three-folds. The first is the dependence on mathematics alone, the second is the dependence on the student alone, and the third is the dependence on mathematics as well as the student. In the first case, skillful uses are made by mathematical textbook writers. The body of material in a textbook should progress according to the cunning of reason combined with the syllogistic analysis, and this is the attitude held in this essay. I shall present many examples of showing this progression. To the second case belong some well-known educations, for instance the Montessori education and the Waldorf education. To the third case belong the mathematics teachers. This dependence is the most delicate and complex one, and can directly influence the result of teaching and learning. The syllogistic analysis and the skillful use of the cunning of reason are considered based on the material in the textbook and the academic discipline that the student is located and the academic ability of the student at proper moment. It is always helpful, at stage of preparing lecture, to carry out a syllogistic analysis in combination with the cunning of reason.

Now I describe my philosophical standpoints. I believe in a large part of mathematics consists in syllogisms and their interrelationships. Two of the most important reasonings in mathematics are the induction and the deduction. The deductive reasoning is nothing but the classical Aristotelean syllogism which consists of a major premise, a minor premise and a conclusion. However the Aristotelean syllogism alone can not produce anything new and can not be used satisfactorily within the inductive reasoning.

Syllogisms play an important role in mathematical practice. Bourbaki for instance tried to construct the whole mathematics by three mother structures, all three structures are defined by syllogisms. Some researchers in philosophy of mathematics use a special syllogism called complementarity in searching for mathematical knowledge. According to Otte, mathematical practice, which has increasingly liberated itself from metaphysical and ontological agendas since Cantor and Hilbert, requires a complementarist approach – perhaps more than any other field of knowledge – in order to be understood properly (2003). The complementarist approach to sets and numbers, advocated by Otte, is parallel to the description theoretical approach to sets and numbers of Frege (1953) (the number refers to the year when the cited work appeared not necessarily when the work was written or published for the first time), but the former has great potential in other domains where the latter fails completely.

Inasmuch as a large part of mathematics consists in syllogisms and their interrelationships, that part of mathematics as an educational task should be organized syllogistically. Only so, when a piece of mathematics is transformed into a piece of educational mathematics, invariance can gain a mean-
ing. The more invariance the transformation can preserve the better educational value it can have.

The philosophical discussion I make in the chapter on the cunning of reason is the result of the inquiry into the question of what common part that to make himself work of Piaget and to master his own behavior of Vygotsky share. The said discussion led me first to the conclusion that the Marxist concept of labour and the Vygotskian concept of mediated activity are dual to each other, and then to the formulation of a model for the cunning of reason which unites them. Simple and usable unifications of known models are one of the aims in this essay.

In the contemporary philosophy of mathematics education, the assertion of the student constructs his own mathematics becomes less and less controversial. The first question then is of what the student can construct beyond syllogisms, the induction and the cunning of reason. If the student constructs mathematics in a classroom, it gives rise to the second question of what is the role the teacher plays in that construction.

As to the first question, I believe that the student can only construct a piece of mathematics from what is presented to him and what he possesses at the moment by syllogisms, the induction and the cunning of reason; as to the second, the teacher plays an important role in the construction by supplying the student with suitable material (the first two sides in a syllogism for instance) and by making skillful uses of the cunning of reason. The construction is carried out piece by piece, and for me, it makes no didactic sense if the student can construct the number five in the sense of Frege but can not construct that number by adding the given numbers two and three. At the level of curriculum and that of class instruction, it is the construction by syllogisms, the induction and the cunning of reason that really matter.

The use of syllogisms in education can be traced all the way back to Plato. Indeed, the Republic (Plato, 1997) is probably the first recorded work ever written on philosophy of education in the modern sense. It not only sets out all those norms which characterize that ideal state but also emphasizes the necessity of education in reaching that state in which a citizen is characterized by being happy. Of course no human is happy at his birth and he has to transform himself from being unhappy into being happy. The transformation can only be achieved through education. Book VII of Republic begins with “Next, I said, compare the effect of education and of the lack of it on our nature to an experience like this:”. And then Plato describes vividly the form of truth the prisoner in a fetter and bond may conclude. The truth consequently depends on the fetter and bond and such truth is not genuine but illusive. In searching for the genuine truth the old fetter and bond must be cracked, but this results in new fetter and bond in which the truth is sublated though not final. Observation made by the prisoner in a fetter and bond is observation in one perspective. The truth, the ultimate
goal of any inquiry, is always embedded in some perspectives and only in
which observation on the truth can be made; and it consists in a progressive
process in which different perspectives pass into each other. Synthesizing
different perspectives (change of perspective) is particularly significant in
the domain of problem solving, for a given problem with a given solution is
nothing but a fetter and bond. Without change of perspective, the problem
solving forces the student to learn mathematics from one perspective often
given by the others. In such a problem solving the student becomes the
prisoner of the one who makes the problem.

Syllogisms in this essay can be further classified into microscopic ones
and macroscopic ones, with the Aristotelean syllogism as the characteristic
example of the former and the necessity/contingency as that of the latter.
A syllogism is macroscopic if the synthesis in it can be almost anything, and
is microscopic if otherwise. The Aristotelean syllogism is microscopic for
the synthesis depends uniquely on the thesis and the antithesis. In fact, a
majority of syllogisms mentioned in the essay are microscopic. A syllogism
behaves like a multiplication in the sense that for the given sides $a$ and $b$ the
third side $c = a \cdot b$ can be obtained by multiplying (synthesizing) $a$ and $b$
together. This model of syllogism unifies many known models, for instance
the Aristotelean syllogism and the mother structures in Bourbaki. The uni-
fication is introduced not for the sake of rhetoric but for a systematical use
in teaching and learning of mathematics. Inasmuch as the models are intro-
duced for purpose of using, they should be simple enough to be understood
and used and general enough to contain some known models. This principle
holds also for other unifications made in the present essay. To wit, the model
for cunning of reason unifies the Marxist concept of labour, the Vygotskian
concept of mediated activity, and several models used by Piaget. The model
for stable syllogism is introduced to unite the Piagetian reversibility and the
formulation given by Vygotsky on the role played by the mediated activity
in transition from a lower process to a higher process.

Now I shall describe in some details the contents of the present essay
which consists of the present introduction and some chapters.

The first chapter is a collection of cases in teaching mathematics and in
Swedish context. The common issue in these cases is interactions between
some pairs of two parts.

The second chapter addresses the issue concerned and the approach used,
not only in relation to itself, but also in relation to others.

The third chapter contains three examples in an educational process.
One example is on natural numbers, the other is on fractions, and the last
is on equation solving. The common part of these three examples is the
underlying syllogistic analysis which is carried on them.

The fourth chapter conducts a general discussion on syllogisms, their
sides, and their role in Bourbaki. The most important part of this chapter is
a discussion on the structural analysis which contains the syllogistic analysis
as its special case. The structuralism formulated by Cherryholmes is an interpretation made by him on the three laws in the Piagetian structuralism.

The fifth chapter tells the story of a unification of body and mind. The purpose of the narrative is to search for a proper role played by the cunning of reason in any syllogistic progression. Here I have found four properties shared by all the examples on the cunning of reason. These four properties become a system of axioms for a model for the cunning of reason. Along the way a duality between the Marxist concept of labour and the Vygotskian thinking is explained.

The sixth chapter contains some comprehensive applications and some elaboration of the contentions enunciated in the earlier chapters.

The seventh chapter discuss some teaching issues in Probability and Statistics within the framework of this essay. Since research on statistics education at university level practically does not exist, the chapter concentrates on three special topics.

The essay ends with a paragraph which contains a short summary on the method, some final words and acknowledgment.
Chapter 1

Some Cases

Some cases will be presented in this chapter, they are called cases because of their arrival from mathematics teaching situations. The purpose is to show certain interactions between two complementary parts existing in them. So long as the interaction is concerned the part of mathematics in these case is operational not ontological. The word interaction appears sometimes in form of synthesis or complementarity or passing into each other or change of perspective and so on.

That mathematics consists in such interactions is a well-known fact. The basic elements of mathematics are logic and intuition, analysis and construction, generality and individuality; it is only the interplay of these antithetic forces and the struggle for their synthesis that constitute the life, usefulness, and supreme value of mathematical science (Courant and Robbins, 1996).

In mathematics education qualitative interactions are not enough, in fact, quantitative interactions are in focus of mathematics teaching. Any quantitative interaction involves two concrete parts and consists in description of interaction in terms of passing into each other or direct construction of each other or in terms of change of perspective.

Case 1. Once I observed a girl of thirteen years old in doing an exercise on the percentages. The exercise is the third one on the page 128 of the textbook (Carlsson et al., 2001). Let me translate the exercise into English.

Martin’s new pair of jeans shrink 4.5% after the first wash. How many cm shorter become the pair of jeans after the first wash if they are 110 cm long before the first wash?

Before she began the task she did the first two exercises on that page. For instance the first exercise is to express some given percentages in terms of numbers such as

\[12\% = 0.12, \quad 23.4\% = 0.234, \quad 9.3\% = 0.093.\]

She did them quickly and accurately.
And then she turned to the third one. A general formulation of the task is to compute \( x\% \) of a quantity \( y \). She did not only the third exercise but also some other exercises of the same sort. I now present her way of computing 4.5\% of 110. It goes like this:

\[
\left( \frac{110}{100} \right) \cdot 4.5 = 1.1 \cdot 4.5 = 4.95.
\]

The computation shows clearly that the girl has knowledge of percentage, and indeed, the ontological knowledge of percentage, that is to divide 110 into 100 equal pieces and to conclude that the amount of 4.5 pieces is 4.5\% of 110.

There is of course the operational knowledge of percentage, and it is this knowledge that is often used in a normal teaching situation and in a daily life. This operational knowledge of 4.5\% of 110 is

\[
0.045 \cdot 110 = 4.95.
\]

Since the girl had the ontological knowledge of percentage, I wanted to know if she also had the operational knowledge of percentage. Thereby I said to her: “there is a quicker way to compute 4.5\% of 110, and it is 0.045 \cdot 110 = 4.95”. She said: “why?”. Then I explained to her that 4.5\% of 110 is (as she did)

\[
\left( \frac{110}{100} \right) \cdot 4.5
\]

and

\[
\left( \frac{110}{100} \right) \cdot 4.5 \equiv \frac{110 \cdot 4.5}{100 \cdot 1} = \frac{110 \cdot 4.5}{100} \cdot \frac{1}{100} = 110 \cdot 0.045 = 0.045 \cdot 110.
\]

After I said this and I took notice of confusion on her face, as if the number on the left side went through that magic box and then became the number on the right side. What this box does in a normal teaching situation is to show a didactic transposition (cf. the beginning of Chapter 2) of the commutativity, as used in (Kang and Kilpatrick, 1992). The box is also an apparatus of synthesizing the ontological knowledge of percentage and the operational knowledge of percentage. A synthesis in a normal teaching situation appears often in form of change of perspective. The change of the ontological perspective to the operational perspective discussed above is facilitated mathematically by a didactic transposition

\[
100 \cdot 1 = 1 \cdot 100.
\]

Mathematically, this is the only way to synthesize those two perspectives.

Of course a change of perspective in mathematics education is a great challenge, and I believe to help the student to change a perspective is the
most important task for the teacher. In our case, I have to discuss the above-mentioned box at several occasions in order to help that girl to change the ontological percentage to the operational percentage. It took me some time but I succeeded in that.

**Case 2.** A few doubt about the fact that children know small numbers such as 1, 2, 3, 4 at very early age. What are the underpinnings of this fact? I believe the answer is a complex, but at the same time I think a special activity is one in the complex. This activity is to copy mentally and physically given things. I have experimented many times with children in asking them to copy (reproduce) simple patterns showed, for example, a pattern such as

```
• • • • • •
```

made of black balls. It does not take long time for the children to construct with white balls the following pattern

```
◦ ◦ ◦
◦ ◦ ◦ ◦ ◦ ◦ ◦ ◦
```

Although the pattern made of white balls can deviate from the pattern made of black balls; most of time, it can be clearly recognized that the latter is constructed upon the former used as a blue-print. Among the children they often copy activities from each other. This leads me to believe that not only the child can construct a pattern from a blue-print but also the child can treat the blue-print as a construction from his own constructed pattern. This bilateral construction of two patterns is extremely important at the most fundamental psychological level. Possession of such ability is a proof on existence of some senses of invariance in the child. The sense of number is one of them.

**Case 3.** A linear equation, say $2x + 4 = 8$, occupies a special place in mathematics education. At a close look it reveals that most of didactical discussion on the equation is not about the way of solving it but about the way of removing psychological unease caused by the symbol $x$. Thus the equation

```
2x + 4 = 8
```

becomes

```
2 \cdot - + 4 = 8
```

or

```
2 \cdot □ + 4 = 8
```
or

\[ 2 \cdot \square + 4 = 8. \]

The symbols \(-\) and \(\square\) are blanks which invite the student to guess, while there is a magical number hidden behind the symbol \(\square\) waiting for the student to uncover. These didactic devices remove certain psychological unease but at the same time bring in logical nonsense, for it makes no sense to talk about the multiplication of these symbols by the number 2. In the equation \(2x + 4 = 8\) however, the letter \(x\) is a number, so that the equation can be manipulated by mathematical means.

Any equation

\[ A = B \]

should be in the first place looked upon as a structure synthesized by two parts \(A\) and \(B\). Solving such an equation mathematically amounts to finding interrelationships among equations of similar sort, in symbols,

\[ A = B \iff C = D. \]

For instance, solving the equation \(2x + 4 = 8\) amounts to finding interrelationships among equations

\[ 2x + 4 = 8 \iff 2x = 4 \iff x = 2. \]

Any equation here is a didactic transposition of the other. The sense of invariance under such didactic transpositions is the sense of solvability.

**Case 4.** Two exercises in a popular Swedish textbook on Analysis are to simplify

\[ (x + 3)(x - 3) - (x + 3)^2 \]

and to factorize

\[ x^2y + 2x^2 - y - 2. \]

They all have something to do the distributivity between the addition and multiplication. The most standard form of the distributivity is

\[ a(b + c) = ab + ac. \]

The standard form may not be the best form in a didactic situation, and thereby needs to be didactically transposed. In fact, the most fundamental didactic transposition of the distributivity is

\[ a(b + c) = ab + ac = a(b + c) \]

which is a synthesis of \(a(b + c) = ab + ac\) and \(ab + ac = a(b + c)\). The first one is an additive expansion (of multiplication) while the second one
is a multiplicative expansion (of addition). Thereby they are the direct constructions of each other through the cunning of reason.

The first exercise is much easier for the student in comparison to the second, for the student reasons as

$$(x + 3)(x - 3) - (x + 3)^2 = x^2 - 9 - (x^2 + 6x + 9) = -6x - 18$$

without much trouble. It seems the student feels at home with $a(b + c) = ab + ac$ and strange to $ab + ac = a(b + c)$. This can only be caused by lacking use of the cunning of reason. When the student feels equally at home with $ab + ac = a(b + c)$, then the second exercise becomes easy. It goes like

$$x^2y + 2x^2 - y - 2 = x^2(y + 2) + (-1)(y + 2) = (x^2 - 1)(y + 2) = (x + 1)(x - 1)(y + 2)$$

or

$$x^2y + 2x^2 - y - 2 = x^2y - y + 2x^2 - 2 = y(x^2 - 1) + 2(x^2 - 1)$$

$$= (x^2 - 1)(y + 2) = (x + 1)(x - 1)(y + 2).$$

**Case 5.** Most of the students I encountered had the knowledge of the formula $(a + b)^2 = a^2 + 2ab + b^2$ but a few had the knowledge of the method of completing a quadrate, let alone the connection between them. Thereby solving a quadratic equation becomes a mechanical use of some ready-made formulas, and understanding quadratic functions consequently stays at surface.

The formula $(a + b)^2 = a^2 + 2ab + b^2$ and the method of completing a quadrate are didactic transpositions of each other. The point of departure of this didactic transposition is a geometric interpretation of that formula. The interpretation is to cut a perfect square into four rectangles as showed in the following figure:

```
+----+----+----+
|    | ab  | b^2 |
+----+----+----+
| a   | a^2 | ab  |
+----+----+----+
| a   | b   |
```

Now let us interpret $x^2 + px$ in a similar fashion. Since $x^2 + px = x^2 + 2x(p/2)$, it can be interpreted as a sum of areas of the three rectangles in the following figure:

```
+----+----+----+
|    | px  |  
+----+----+----+
| x   | x^2 | px  |
+----+----+----+
| x   | p/2 |
```
which is clearly not a perfect square. To complete a square means therefore to add another square with side $p/2$ into the preceding figure, so we get the following figure:

\[
\begin{array}{c|c|c}
\frac{p}{2} & \frac{px}{2} & \cdots \\
\hline
x & x^2 & \frac{px}{2} \\
\hline
x & \frac{p}{2} & \\
\end{array}
\]

and the figure shows clearly

\[x^2 + px = \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2\]

which is the method of completing a quadrate.

This case therefore emphasizes the interaction between the formula $(a + b)^2 = a^2 + 2ab + b^2$, also called the quadratic rule, and the method of completing a quadrate, in symbols,

\[
\text{the quadratic rule } \longleftrightarrow \text{ the method of completing a quadrate}
\]

**Case 6.** Teaching of negative numbers in schools appeals completely to metaphors. Standard metaphors introduced for the purpose of facilitating understanding of negative numbers give rise to certain hidden confusions on positive numbers, which are supposed to be well understood by the child at the time when he is exposed to negative numbers. All metaphors designed for teaching negative numbers are in principle of the same kind. They depend on the way quantities in possession and quantities in debt shall be interpreted.

Before metaphors on negative numbers are introduced to the child, a positive number, say the number 5, is neutral to the child. That is the child can supply the number 5 with some unit, say book, but the 5 books are never understood as the 5 books in the child’s possession. The question now is how to express the fact that the 5 books are in the child’s possession, if $−5$ books mean the 5 books is in the child’s debt. A natural candidate is $+5$ books which mean the 5 books are in the child’s possession. However I do not think this good enough.

The neutral 5 books, neither in the child’s possession nor in the child’s debt, mean a collection of five books, meaning 5 times 1 book. If you claim the 5 books are in your possession, you of course mean each one in that collection is in your possession. Hence the most natural way to define 5 books in possession is $5 \cdot (+1)$ here $+1$ means one book in possession. Completely analogous to this, 5 books in debt is $5 \cdot (−1)$ here $−1$ means one book in debt. Hence it is natural define

\[5 = 5 \cdot (+1), \quad -5 = 5 \cdot (-1)\]
and a relation between +1 and −1 in such a fashion that

\[(+1) + (−1) = 0.\]

Such a definition of negative numbers makes addition very natural. Since −5 = 5 · (−1) and −7 = 7 · (−1), it follows that

\[-5 + (−7) = 5 · (−1) + 7 · (−1) = 12 · (−1) = −12.\]

Subtraction of two negative numbers is much more difficult to handle, for a thing such as −(−2) appears.

First of all, it would be an illusion to think the number −(−2) has an ontological meaning. In fact it is essential to point out that an educational meaning of −(−2) must be operational.

Now that we know the meaning of −2, the most natural meaning we can give to −(−2) is (−1) · (−2). Now it is the turn of the distributivity. Since

\[-2 + 2 = 2 · (−1) + 2 · (+1) = 2 · ((−1) + (+1)) = 2 · 0 = 0,\]

and since

\[-2 + (−(−2)) = 2 · (−1) + (−1) · (−2) = (−1) · (2 + (−2)) = (−1) · 0 = 0,\]

therefore

\[-(−2) = 2,\]

for both −(−2) and 2 can make 2 in debt annihilated.

The construction of −(−2) made above is a global construction, it depends not only on numbers but also on the three fundamental laws of commutativity, associativity and distributivity. However the very heart of this case is a syllogism

\[
\begin{array}{c}
5 · (−1) \\
\downarrow
\end{array}
\leftarrow
\begin{array}{c}
5 \\
\uparrow
\end{array}
\rightarrow
\begin{array}{c}
5 · (+1)
\end{array}.
\]

**Case 7.** The partial integration is a method transforming one integral into another. The aim of the method is not to express an integral by elementary functions with one single stroke. The method is effective only if it is used with care, for otherwise, the method can transform a simple integral into a complex one. If the symbol ↑ means integration while ↓ differentiation then the method of partial integration

\[
\int f(x)g(x) \, dx = F(x)g(x) - \int F(x)g'(x) \, dx
\]

\[
\uparrow \bigg/ \downarrow
\]

\[
F(x) g'(x)
\]
transforms the integral on the left side into that on the right side. Most of students believe a mathematical error is committed if the method of partial integration they perform does not lead to a more simple integral. It is therefore important in teaching to point out that a mathematically correct performance of partial integration can indeed lead to a more complex integral. A genuine understanding of the method of partial integration lies consequently in comparison between two different ways of performing the method.

Let us take a closer look at the integral \( \int x \cdot e^{-x} \, dx \). One way of performing the method of partial integration is

\[
\int x \cdot e^{-x} \, dx = \frac{x^2}{2} \cdot e^{-x} - \int \frac{x^2}{2} \cdot (-e^{-x}) \, dx
\]

and the other is

\[
\int x \cdot e^{-x} \, dx = x \cdot (-e^{-x}) - \int 1 \cdot (-e^{-x}) \, dx.
\]

Now the student has to compare the following three integrals

\[
\int x \cdot e^{-x} \, dx, \quad \int \frac{x^2}{2} \cdot (-e^{-x}) \, dx, \quad \int 1 \cdot (-e^{-x}) \, dx
\]

and to point out the simple one and the complex one and the one in middle. This step is a part of didactic partial integration though not a part of mathematical partial integration.

**Case 8.** In Chevallard’s theory of didactic transposition, conditions and constraints are two fundamental concepts; the former tell what to do whereas the latter what to avoid. Under the condition of re-writing an expression such as \( 2 \cdot 5 + 5 \), many didactic transpositions can be made, for instance,

\[
2 \cdot 5 + 5 = x + 2 \cdot 5 + 5 - x = 2 \cdot 5 + 7 - 2 = 5 + 5 = 10.
\]

Under the constraint of computing the expression \( 2 \cdot 5 + 5 \), many didactic transpositions should be avoided; in the example above, the one

\[
2 \cdot 5 + 5 = 5 + 5 = 10
\]

is allowed while the others should be avoided.

By the time when the students begin with the study of derivatives, they are well equipped with the knowledge on the four operations and \((x+h)^2\) and so on. Under the condition of removing the denominator in the expression
(5(x + h)^2 - 5x^2)/h, most of the students can come up with the following didactic transposition

\[
\frac{5(x + h)^2 - 5x^2}{h} = 10x + 5h.
\]

The derivative \( f'(x) \) of a function \( f(x) \) is

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.
\]

Hence any didactic transposition of derivative is an interaction between the condition \( (f(x+h) - f(x))/h \) and the constraint \( \lim_{h \to 0} \). In computing \( f'(x) \) where \( f(x) = 5x^2 \) the said interaction is of two kinds, the first is

\[
\frac{f(x + h) - f(x)}{h} = \frac{5(x + h)^2 - 5x^2}{h} = \frac{5(x^2 + 2xh + h^2) - 5x^2}{h} = \frac{10xh + 5h^2}{h} = 10x + 5h \\
\to 10x = f'(x)
\]

as \( h \to 0 \) and the second is

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{5(x + h)^2 - 5x^2}{h} = \lim_{h \to 0} \frac{5(x^2 + 2xh + h^2) - 5x^2}{h} = \lim_{h \to 0} \frac{10xh + 5h^2}{h} = \lim_{h \to 0} 10x + 5h = 10x.
\]

Only at the late stage in the process of acquiring derivative, the students feel at home with the both didactic transpositions. At the early stage, the first transposition has great advantage over the second, for it lies nearly entirely within the familiar territory of the students. This case emphasizes the interaction between conditions and constraints, in symbols,

\[
\text{condition } \longleftrightarrow \text{ constraint}
\]
Case 9. In decomposition (expansion) of a rational function into a sum of partial fractions a premise is invoked, for instance, in decomposing the rational function

\[
\frac{4x^2 - 1}{(x - 2)^3},
\]

the premise is

\[
\frac{4x^2 - 1}{(x - 2)^3} = \frac{A}{x - 2} + \frac{B}{(x - 2)^2} + \frac{C}{(x - 2)^3}
\]

for three numbers \(A, B, C\) to be determined. In teaching such a decomposition, a question occurs repeatedly, why three numbers (in the preceding example)?

A premise is just like an algorithm; and in teaching an algorithm the emphasis is on the use of some ready-made steps not so much on understanding them. The division algorithm in dividing say 83 by 5 and the Euclid’s algorithm in determining the greatest common divisor of two given numbers are of typical.

The existence of algorithms in education has good reason, it is neither necessary nor practical to know the deduction behind the algorithms. This however does not mean we should altogether ignore any explanation on them to the students. In fact some partial explanation can be a good teaching point, and heuristic explanation is one such.

One heuristic reasoning common to entire mathematics is the counting principle stating that in determining \(n\) quantities \(n\) equations are needed. This principle is simple to use and offers the first testing ground for things to be done.

Let us take a closer look at the example mentioned above. The numerator of that rational function, also the left side of the premise is

\[
4x^2 - 1.
\]

The right side of the premise is

\[
\frac{A}{x - 2} + \frac{B}{(x - 2)^2} + \frac{C}{(x - 2)^3} = \frac{A(x - 2)^2 + B(x - 2) + C}{(x - 2)^3}
\]

and its numerator is

\[
A(x - 2)^2 + B(x - 2) + C.
\]

These two numerators are both polynomials of degree at most two, and such polynomials have three coefficients. By equating the coefficients to \(x^2, x^1\) and \(x^0\) respectively, three equations are obtained, therefore three numbers can be determined. This also explains the reason behind the requirement
that the degree of the numerator in the rational function to be decomposed must be strictly less than that of denominator.

The interaction

\[
\text{algorithm} \leftrightarrow \text{heuristics}
\]

has great an educational value.

**Case 10.** The perpendicularity is an intuitive idea and the orthogonality is a formalized idea. I do not remember the number of times when I was asked to explain the reason that two vectors \( v_1 = (x_1, y_1) \) and \( v_2 = (x_2, y_2) \) are orthogonal if their inner product \( x_1x_2 + y_1y_2 \) is zero. The orthogonality is a concept in Linear Algebra, and is constructed originally from the perpendicularity in Plane Geometry. In fact these two concepts can be constructed from each other, and the construction has a great educational value.

It all starts with the Pythagorean theorem, and it states that the three sides in a triangle

\[
\begin{array}{c}
\text{b} \\
\text{c} \\
\text{a}
\end{array}
\]

in which two sides with lengths \( a \) and \( b \) are perpendicular satisfy the equation

\[
a^2 + b^2 = c^2.
\]

The less well-known is the inverse of this theorem which states that if three sides in a triangle satisfy the preceding equation then the sides with the lengths \( a \) and \( b \) must be perpendicular. Since the proof of this inverse theorem is not easy to find in modern textbooks, I shall offer here a such.

I shall give an indirect proof, that is I assume three sides \( a, b, c \) in a triangle satisfy the equation \( a^2 + b^2 = c^2 \), and further I assume the side with the length \( a \) is not perpendicular to the side with the length \( b \). Then I shall construct a contradiction.

Since the side with the length \( a \) is not perpendicular to the side with the length \( b \), then we could have only the following two options:

\[
\begin{array}{c}
\text{b} \\
\text{c} \\
\text{a}
\end{array} \quad \begin{array}{c}
\text{b} \\
\text{c} \\
\text{a}
\end{array}
\]

I shall construct a contradiction from for instance the left figure. In order to do that, I shall draw a help line which is perpendicular to the side with the length \( a \) as showed below
We have $a_1 \neq 0$, $a_2 \neq 0$ and $a_1 + a_2 = a$. It follows from the Pythagorean theorem that

$$a_1^2 + h^2 = b^2, \quad a_2^2 + h^2 = c^2.$$ 

The second equation together with the assumption that $a^2 + b^2 = c^2$ show

$$a_2^2 + h^2 = a^2 + b^2 = (a_1 + a_2)^2 + b^2 = a_1^2 + 2a_1a_2 + a_2^2 + b^2$$

which gives, since $a_1^2 + h^2 = b^2$,

$$h^2 = a_1^2 + 2a_1a_2 + b^2 = a_1^2 + 2a_1a_2 + a_1^2 + h^2,$$

so that

$$0 = 2a_1^2 + 2a_1a_2 = 2a_1a > 0.$$ 

A contradiction has thereby been constructed, and hence the proof of the inverse of the Pythagorean theorem is complete.

For a vector $v = (x, y)$, let us draw

Thus the length $\|v\|$ of the vector $v = (x, y)$ is $\sqrt{x^2 + y^2}$.

For two vectors $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$, let us draw

It follows from the Pythagorean theorem together with its inverse that $v_1$ is perpendicular to $v_2$ if and only if

$$\|v_1\|^2 + \|v_2\|^2 = \|v_2 - v_1\|^2.$$
Since \( v_2 - v_1 = (x_2 - x_1, y_2 - y_1) \), \( v_1 \) and \( v_2 \) are perpendicular if and only if
\[
x_1^2 + y_1^2 + x_2^2 + y_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 = x_2^2 - 2x_1x_2 + x_1^2 + y_2^2 - 2y_1y_2 + y_1^2
\]
which is
\[
x_1x_2 + y_1y_2 = 0.
\]
This is the motivation in Linear Algebra for defining orthogonality by annihilation of the inner product. This case emphasizes the interaction

\[
\begin{array}{c|c}
\text{Geometry} & \text{Algebra} \\
\hline
\end{array}
\]

for a deep understanding of the perpendicularity and the orthogonality.

**Case 11.** A finite geometric series is a sum of the form
\[
1 + q + q^2 + q^3 + \cdots + q^n
\]
while an infinite geometric series is a sum of the form
\[
1 + q + q^2 + q^3 + \cdots
\]
with \( q \) being called the common quotient in both cases.

In teaching the geometric series the emphasis is on some short expressions for these series. There are many ways that can lead geometric series to the expressions, of which nearly all have a formalized character. It seems very difficult to supply these deductions with intuitive ideas. Let me first of all present two standard ways of handling a finite geometric series. Both ways start with calling the series \( x \), that is, let
\[
x = 1 + q + q^2 + q^3 + \cdots + q^n.
\]
The first way is to multiply the preceding equation by \( q \) first and then to subtract the result from the equation, this gives rise to
\[
x - xq = (1 + q + q^2 + q^3 + \cdots + q^n) - (q + q^2 + q^3 + \cdots + q^{n+1}) = 1 - q^{n+1}
\]
which gives
\[
1 + q + q^2 + q^3 + \cdots + q^n = \begin{cases} 
\frac{1-q^{n+1}}{1-q}, & q \neq 1, \\
n+1, & q = 1.
\end{cases}
\]
And the second way is to write
\[
x = 1 + q + q^2 + q^3 + \cdots + q^n = 1 - q^{n+1} + q(1 + q + q^2 + q^3 + \cdots + q^n)
\]
which is the same as
\[
x = 1 - q^{n+1} + qx.
\]
The last equation leads again to the above-mentioned expression.

In dealing with an infinite geometric series, the assumption of $|q| < 1$ has to be made, in which case,

$$1 + q + q^2 + q^3 + \cdots = \frac{1}{1 - q}.$$ 

Most of the students whom I taught thought of the deduction simple, and at the same time they neither can deduce the procedure themselves nor even remember the result for a long period of time. The principal reason is perhaps that the formal deduction does not involve intuitive ideas. Intuition is not necessarily geometric intuition, and sometimes probabilistic intuition is a good assist to understand a piece of formal deduction. In the past few years I started to combine the above formal deduction with a particular probabilistic model. There is a clear improvement in the students’ understanding geometric series.

The model I use is what is called the fft-distribution (for first time) in statistics. There is a certain game you want to play. Each time the probability you win the game is $p$ and the probability you loose the game is $q$. Of course $p + q = 1$. Your intuition indicates you shall win sometimes if you just keep playing, but you do not know when you are going to win. You also set off a principle for yourself, stop playing immediately after you win for the first time.

If you are very lucky then you win the first time you play the game, in symbols, $W$, the probability for that is just $p$. It could be true that you loose the first game and win the second, in symbols, $LW$, and the probability for that is $qp$. It could be true that you lose the first two game and win the third, in symbols, $LLW$, and the probability for that is $q^2p$. It could be true that you loose the first three game and win the fourth, in symbols, $LLLW$, and the probability for that is $q^3p$. This simple intuitive deduction quickly leads to

$$p + qp + q^2p + q^3p + \cdots = 1.$$ 

This in turn leads directly to

$$1 + q + q^2 + q^3 + \cdots = \frac{1}{p} = \frac{1}{1 - q}.$$ 

This case shows the importance of interaction between formalization and intuition, in symbols,

$$\text{formalization} \leftrightarrow \text{intuition}.$$ 

**Case 12.** Theory of Probability is constructed with help of different models. The underpinnings of the mathematical model are a set $\Omega$ together with some prescribed subsets $A, B, C, \cdots$ and some numbers $P(A), P(B), P(C), \cdots$ attached on them. The beginning of the model is three properties sorted out by Andrei Kolmogorov in 1930s. They are
(P1) \(0 \leq P(A) \leq 1\),
(P2) \(P(\Omega) = 1\),
(P3) \(P(A \cup B) = P(A) + P(B)\) if \(A \cap B = \emptyset\).

It is already a mathematical wonder that such a thin beginning can support a huge body of knowledge including Probability, Statistics, Theory of Stochastic Processes and Quantum Mechanics.

The third property has a clear tendency to treat \(\cup\) as an addition with respect to \(P\). This further leads to another tendency to treat \(\cap\) as a multiplication also with respect to \(P\). This means \(P(A \cap B) = P(A)P(B)\). Unfortunately for most of \(A\) and \(B\) this is not true. However the soundness of it must represent some fine mathematical structure, thus if

\[ P(A \cap B) = P(A)P(B) \]

is true then \(A\) and \(B\) are said to be independent. This definition of independence lies within the mathematical model mentioned above. The mathematical definition of independence is easy to define but very difficult to handle. There is not much students can test about the mathematical independence in the entire Blom’s book (1989) except for a few simple exercises, for instance to show \(A^*\) and \(B^*\) are independent if \(A\) and \(B\) are independent.

Contrast to this, the practical sense of independence offers students plenty of opportunities to test this piece of knowledge. So long as the practical independence is an important area for students to test their knowledge, it must clearly point out there are two functionally different kinds of independence existing in educational Probability. They are

\[ P(A \cap B) = P(A)P(B) \implies A \text{ and } B \text{ are mathematically independent} \]

and

\[ P(A \cap B) = P(A)P(B) \iff A \text{ and } B \text{ are practically independent} \]

In what follows I shall give two examples of each kind.

Assume that two events \(A\) and \(B\) are (mathematically) independent. Then we can prove that \(A^*\) and \(B^*\) are also (mathematically) independent. Since \(A\) and \(B\) are independent, then \(P(A \cap B) = P(A)P(B)\), so that

\[ P(A^* \cap B^*) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B) \]
\[ = 1 - P(A) - P(B) + P(A)P(B) \]
\[ = (1 - P(A))(1 - P(B)) \]
\[ = P(A^*)P(B^*). \]

Hence \(A^*\) and \(B^*\) are (mathematically) independent.

Assume that two persons \(A_1\) and \(B_1\) bought (practically) independently of each other two lottery tickets. Let \(A\) be the event that \(A_1\) won something
and $B$ the event $B_1$ won something. Assume in addition that $P(A) = p_1$ and $P(B) = p_2$. Then we know for sure that $P(A \cap B) = p_1 p_2$.

This case shows an interrelationship among

practical independence
\[ P(A \cap B) = P(A)P(B) \]

mathematical independence

in statistics education.

**Case 13.** The Swedish version of the textbook (Blom, 1989) is a classic on the theory of probability and statistics. One of the exercises on the page 38 is about the conditional probability.

$A$ and $B$ have eleven fruits, three of which are poisonous. $A$ eats four fruits and $B$ six, all chosen at random; their pig gets the remaining fruit. Determine:

(a) the probability that the pig is not poisoned;

(b) the conditional probability that $A$ and $B$ are both poisoned given that the pig is not poisoned;

(c) the probability that $A$ and $B$ are both poisoned and the pig is not poisoned.

Let $A_1$ be the event that $A$ is poisoned, $B_1$ the event that $B$ is poisoned, and let $C$ be the event that the pig is not poisoned. It is very clear that $P(C) = \frac{8}{11}$. Since

\[ P(A_1 \cap B_1 | C) = \frac{P(A_1 \cap B_1 \cap C)}{P(C)} \]

so $P(A_1 \cap B_1 \cap C) = 8 P(A_1 \cap B_1 | C) / 11$. Hence the most interesting part of this exercise is to determine the conditional probability $P(A_1 \cap B_1 | C)$.

A determination of this conditional probability involves a fundamental doctrine of the theory of probability, that is the equivalence of different random models. The equivalence of sets in terms of the one-to-one correspondence is the genesis of the sense of number, while the equivalence of random models is the genesis of the sense of probability. In the case of number, the sense of number is preconditioned by the concept of being equal in quantity, while the sense of probability is preconditioned by the concept of being equal in probability. Consequently this last mentioned concept is no doubt the most fundamental one in the entire educational probability and statistics.

In the random model of the eleven fruits, let us determine the conditional probability $P(A_1 \cap B_1 | C)$. We have to determine $P(A_1 \cap B_1 \cap C)$ first. The total number of different ways of eleven fruits are eaten in such a fashion that the dog eats one, $A$ eats four and $B$ eats six is

\[ \binom{11}{1} \cdot \binom{10}{4} \cdot \binom{6}{6} = 11 \cdot \frac{10!}{4! \cdot 6!} \cdot 1 = 11 \cdot 7 \cdot 3 \cdot 10. \]
The event $A_1 \cap B_1 \cap C$ means that either the dog eats one of the eight good fruits and $A$ eats one of the three bad fruits in which case $B$ eats two of the three bad fruits, or else the dog eats one of the eight good fruits and $A$ eats two of the three bad fruits in which case $B$ eats one of the three bad fruits. The total number of such (favourable) ways of eating is

\[
\binom{8}{1} \cdot \binom{3}{1} \cdot \binom{7}{3} \cdot \binom{6}{6} + \binom{8}{1} \cdot \binom{3}{2} \cdot \binom{7}{2} \cdot \binom{6}{6} = 8 \cdot 3 \cdot 5 \cdot 7 + 8 \cdot 3 \cdot 3 \cdot 7.
\]

Hence

\[
P(A_1 \cap B_1 \cap C) = \frac{8 \cdot 3 \cdot 8 \cdot 7}{11 \cdot 7 \cdot 3 \cdot 10} = \frac{4 \cdot 8}{11 \cdot 5},
\]

so that

\[
P(A_1 \cap B_1 | C) = \frac{P(A_1 \cap B_1 \cap C)}{P(C)} = \frac{4 \cdot 8}{11 \cdot 5} / \frac{8}{11} = \frac{4}{5}.
\]

This conditional probability can be determined by yet another random model made of ten fruits, three of which are poisonous. With this model

\[
P(A_1 \cap B_1 | C) = P(A_1 \cap B_1)
\]

since the dog has eaten one of good fruits. We know that

\[
P(A_1 \cap B_1) = 1 - P(A_1^* \cup B_1^*) = 1 - P(A_1^*) - P(B_1^*) + P(A_1^* \cap B_1^*)
\]

and $P(A_1^* \cap B_1^*) = 0$. We need now to determine $P(A_1^*)$ and $P(B_1^*)$. The determination can be similar to that just used or can be made by the famous urn-model (Blom, 1989, p. 18). By this model

\[
P(A_1^*) = \frac{\binom{7}{4} \cdot \binom{3}{0}}{\binom{10}{4}} = \frac{7!}{4! \cdot 3!} / \frac{4! \cdot 6!}{6!} = \frac{1}{6}
\]

and

\[
P(B_1^*) = \frac{\binom{7}{6} \cdot \binom{3}{0}}{\binom{10}{6}} = \frac{7!}{6! \cdot 1!} / \frac{6! \cdot 4!}{4!} = \frac{1}{30}.
\]

Together it follows that

\[
P(A_1 \cap B_1) = 1 - \frac{1}{6} - \frac{1}{30} = \frac{24}{30} = \frac{4}{5},
\]

a result obtained once by another random model.

Random models of being equal in probability are used in one way or another throughout Probability and Statistics, and such models are underpinnings of the probabilistic intuition. This case emphasizes the importance of interaction among different random models and the concept of being equal in probability in the interaction.
Case 14. The four arithmetic operations have a striking analogue in the theory of probability. The operation $\cup$ is similar to the addition in the sense of

$$P(A \cup B) = P(A) + P(B)$$

if $A \cap B = \emptyset$. The operation $\setminus$ is similar to the subtraction in the sense of

$$P(A^* ) = P(\Omega \setminus A) = P(\Omega) - P(A) = 1 - P(A).$$

The operation $\cap$ is similar to the multiplication in the sense of

$$P(A \cap B) = P(A)P(B)$$

if $A$ and $B$ are independent.

Now the conditional probability $P(\cdot | \cdot)$ is similar to the division in the sense of Total Probability Theorem. In the simplest non-trivial case of $\Omega = A \cup B$ with $A \cap B = \emptyset$, this theorem states

$$P(C) = P(C|A) \cdot P(A) + P(C|B) \cdot P(B),$$

which can be written as

$$P(C \cup C) = P(C|A) \cdot P(A) + P(C|B) \cdot P(B)$$

for $C \cup C = C$. Thus it becomes similar to an arithmetic formula

$$2c = \frac{c}{a} \cdot a + \frac{c}{b} \cdot b.$$

I shall now illustrate a use of this theorem by an exercise on the page 38 of (Blom, 1989).

From a signpost marked MIAMI two randomly chosen letters fall down. A friendly illiterate puts the letters back. Determine the probability that the signpost again reads MIAMI.

Let $A$ be the event that two randomly chosen different letters fall down, and let $B = A^*$. Let $C$ be the event that the signpost reads again MIAMI. It is clear that $P(C|A) = 1/2$ and $P(C|B) = 1$. In addition,

$$P(B) = 2 \sqrt{\binom{5}{2}} = \frac{2 \sqrt{10}}{5} = \frac{1}{5}$$

and therefore $P(A) = 4/5$. Together it follows that

$$P(C) = P(C|A) \cdot P(A) + P(C|B) \cdot P(B) = \frac{1}{2} \cdot \frac{4}{5} + 1 \cdot \frac{1}{5} = \frac{3}{5}.$$

The additive property (P3), the independence and Total Probability Theorem thus describe an interaction between Probability on the one side and Arithmetic on the other side, in symbols,

\[
\text{Probability} \leftrightarrow \text{Arithmetic}
\]
All these cases come from real teaching situations, and their levels range from elementary to high. Each case exhibits a teaching point (where the difficulty of learning is located), such a point depends of course on a teaching situation. A common part of these teaching points lies in a syllogism (a box in some cases), for there lies the difficulty of learning as well. In some cases the syllogism can also be interpreted as the change of perspective. Although some of the above teaching situations might be modified in such a way that the syllogism is less visible, but it is undeniable that it has a function of uniting the teaching points. Hence the syllogism should have a proper position in mathematics education.
Mathematics education, as an academic discipline, had existed three decades or so by the time when Niss (1999) wrote that survey on the subject. Mathematics education has a dual nature. On the one side it offers something to the student, and on the other side it is a field of research in which this ‘something’ is the subject. Since a majority of research mathematicians are university teachers as well, mathematics education should be of interest to both mathematics educators and research mathematicians. Niss indicates the mathematics teacher should devise innovative formats and ways of teaching or new kinds of student activity, should have a proper selection and sequencing of the material to be taught, and should bring in the clarity and brilliance into his presentation. All these fall into the category of the effective improvements of our modes of teaching. The category is bigger than what has been said, in fact, it also contains the better understanding of the epistemological paths leading to mathematical knowledge, insight and ability, and the obstacles that may block these paths. Since mathematical knowledge, insight and ability are parts of mathematics itself, the effective improvements on our teaching improves us, as research mathematicians, as well. However, the over-arching, ultimate end of the whole enterprise is to promote/improve students’ learning of mathematics and acquisition of mathematical competencies. (Niss, 1999)

There are many ways to approach this end, and the approaches certainly depend on circumstances. To transform mathematical knowledge from a textbook or from heads of some mathematics researchers into something in a notebook of some teacher who shall teach this piece of mathematical knowledge to some students is one of the approaches. The transition from knowledge regarded as a tool to be put to use, to knowledge as something to be taught and learnt, is the didactic transposition of knowledge (Chevallard, 1988). The theory of didactic transposition created by Chevallard (1985) is
for the moment an active and important research field within mathematics education, and the present essay fits well in this field.

The theory of didactic transposition was initiated by the theory of didactical situation (Brousseau, 1997). Contrast to a didactic transposition, a didactical situation is much more difficult to define, to wit,

The modern conception of teaching therefore requires the teacher to provoke the expected adaptation in her students by a judicious choice of “problems” that she puts before them. These problems, chosen in such a way that students can accept them, must make the students act, speak, think, and evolve by their own motivation. Between the moment the student accepts the problem as if it were her own and the moment when she produces her answer, the teacher refrains from interfering and suggesting the knowledge that she wants to see appear. The student knows very well that the problem was chosen to help her acquire a new piece of knowledge, but she must also know that this knowledge is entirely justified by the internal logic of the situation and that she can construct it without appealing to didactical reasoning. Not only can she do it, but she must do it because she will have truly acquired this knowledge only when she is able to put it to use by herself in situations which she will come across outside any teaching context and in the absence of any intentional direction. Such a situation is called an adidactical situation. Each item of knowledge can be characterized by a (or some) adidactical situation(s) which preserve(s) meaning; we shall call this a fundamental situation. But the student cannot solve any adidactical situation immediately; the teacher contrives one which the student can handle. These adidactical situations arranged with didactical purpose determine the knowledge taught at a given moment and the particular meaning that this knowledge is going to have because of the restrictions and deformations thus brought to the fundamental situation.

This situation or problem chosen by the teacher is an essential part of the broader situation in which the teacher seeks to devolve to the student an adidactical situation which provides her with the most independent and most fruitful interaction possible. For this purpose, according to the case, the teacher either communicates or refrains from communicating information, questions, teaching methods, heuristics, etc. She is thus involved in a game with the system of interaction of the student with the problems she gives her. This game, or broader situation, is the didactical situation (Brousseau, 1997, p. 30–31).

Thereby, both the theory of didactic transposition and the theory of didactical situation take a completely new philosophical positioning. The most fundamental philosophical doctrine in both theories is this, knowledge has different forms in a didactical process, and each form has a didactical value attached. The teachability of a piece of knowledge is graded by the value of the form in which that piece of knowledge is transposed into. One such example is exhibited by Kang and Kilpatrick (1992, p. 4) in which several transposed forms of the distributive law are given. The example clearly shows that a piece of knowledge is fragile and should be taken care with vigilance. The fragility of knowledge has a paradoxical aspect in that the more we adhere to a specific form, the more we lose the original meaning
in the form. Recording knowledge in a book is often seen as the most efficient way to preserve and transmit knowledge, but that simply increases the need for the teacher to exercise vigilance (Kang and Kilpatrick, 1992, p. 3). The fragility of knowledge is discussed substantially by Brousseau and Otte (1991). The fragility of knowledge forces us to value didactical judgments anew, for it can not live side by side with the absoluteness of such a judgment. Indeed a didactical judgment must be referred to a triad of the teacher, the student, and the knowledge which constitute a ternary relation, the so-called didactic relation (Chevallard, 1988). Such a relation, in symbols, is

\[ f(T, S, K) = 0. \]

In this relation, there are other implicit variables, administration, budget for instance. All these implicit variables are collectively called institutional variables. An example on knowing French based on the didactic relation is described by Chevallard (1988). Explicit form of a didactic relation can not be sought, for it is extremely complicated. In order to determine that form satisfactorily, some principles are needed, for instance, social contract, didactic intent, conditions and constraints, and so on. The didactics of mathematics is, in essence, concerned with teaching of mathematics, that is a process by which the student who does not know some knowledge will be made to learn, and thereby come to know it. The knowledge in such a process must be teachable. The first step in establishing some body of knowledge as teachable knowledge therefore consists in making it into an organized and more or less integrated whole. For many bodies of knowledge taught at school the integrated whole required existed outside school (Chevallard, 1988). An attempt is made in this essay to didactically transpose the integrated whole required existed outside school into the integrated whole organized by syllogisms used inside school. The theory of didactic transposition has been used as a theoretical framework in some theses. For instance, Kang studied the didactic transposition in American textbooks on elementary algebra (1990), and Wozniak studied the conditions and constraints in the statistics education in France (2005).

Mathematical knowledge in mathematics education can be either a piece of knowledge or a body of knowledge. No matter what, each can manifest itself in different forms. Each form has its own didactic value. The aim of a didactic transposition is to raise the didactic value, which is also the main issue of the present essay. A didactic value depends on the didactic relation, thereby can not be investigated on. Hence the first premise for a general research on didactic transposition is the concept of average didactic relation which is a kind of pseudo-didactic relation. In this essay I shall take the average didactic relation as the didactic relation in a teaching free context, for instance, the didactic relation in the head of a textbook author.

Under this didactic relation, the principle of raising didactic value is
the syllogism. I shall call the principle the principle of syllogism. The principle of syllogism consists of two parts — the syllogistic analysis and the cunning of reason. I defend this principle from three perspectives — relevancy, consistency and generality. My defence on relevancy is by showing a epistemological role the syllogism plays in mathematics, philosophy and psychology. My defence on consistency is by using the syllogism consistently throughout. My defence on generality is by organizing a large of amount of mathematics material, both in a piece and in a body, by the syllogism. The defence is the main body of this essay and I hope I defend it well.

The didactical phenomenology of Freudenthal (1983) is intimately related to the theory of didactic transposition. In the first place the book contains many valuable didactic transpositions, and in the second place it has its own philosophical principle. I begin with some quotations from the book.

Phenomenology of a mathematical concept, a mathematical structure, or a mathematical ideal means, in my terminology, describing this nooumenon in its relation to the phainomena of which it is the means of organizing, indicating which phenomena it is created to organise, and to which it can be extended, how it acts upon these phenomena as a means of organizing, and with what power over these phenomena it endows us (1983, p. 28).

What a didactical phenomenology can do is to prepare the converse approach: starting from those phenomena that beg to be organized and from that starting point teaching the learner to manipulate these means of organizing (1983, p. 32).

For this converse approach I have avoided the term concept attainment intentionally. Instead I speak of the constitution of mental objects, which in my view precedes concept attainment and which can be highly effective even if it is not followed by concept attainment (1983, p. 33).

Here comes my interpretation. Objects in our mental world (noumena) are of two kinds, the one kind is the object that can come from others (the object of others) and the other kind is the object that can only come from itself (the object of self). Dually, objects in the real world are also of two kinds, the one kind is the object that can come in others (the object in others) and the other kind is the object that can only come in itself (the object in self), thereby the real world is also the world of phenomena (or the world of events). When a noumenon comes from others, it becomes a concept. Between the plane of noumena and the plane of phenomena, there is another plane of constitution. The didactical phenomenology is thus the design and analysis of ‘homomorphisms’ $T$ and $L$ as in the following diagram:

```
concept | T     | L     |
noumena  | constitution | phenomena
```
Thus Freudenthal quantifies the famous qualitative contention of Kant: “Thus all human cognition begins with intuitions, goes from there to concepts, and ends with ideas.” (1998, p. 622). The phenomenological approach to mathematics education is, under its own principle and under the average didactic relation, to generate valuable didactic transpositions.

The didactical phenomenological model is a global approach to mathematics education from mathematics itself, it has its own philosophical doctrines and method, it treats mathematics as one integrated piece, it strives after interrelationships among local structures from didactical perspective so as to improve students’ learning of mathematics, it plays a very important role in textbook writing, lecture preparation, and class instruction. To this kind of global approach belong (Polya, 1954), (Stolyar, 1974), (Shi, 2008-2010), and the present essay.

Above, I described the main issue and the approach of the essay, and their relation to others. Below, I shall in short say something on the syllogism which is the main tool of the essay.

A local structure in the present essay is a syllogism. A syllogism is a binary reasoning, that is the end result depends on two prescribed sides. If the end result depends on only one prescribed side, then we have a unary reasoning; and if the end result depends on three given sides, then we have a ternary reasoning. A deductive reasoning is binary, if the number of sides is more than two then the reasoning is also called the inductive reasoning. In this essay, a unitary reasoning is a unary reasoning combined with the cunning of reason. The use of macroscopic syllogisms in epistemology is a classical issue, and it consists in dualism, with Descartes’ duality between mind and body as a typical example. This epistemological approach solves some paradoxes but at the same time brings in profound mysticism. In the section on Frege, I examine the use of the duality between *not identical* and *identical* in defining the number zero and number one. I use this example as a sign of warning for showing what should be avoided in a didactical process. The binary syllogism in the present essay is in fact a very special one (for the details see Chapter 3), it has the form of

\[
\begin{align*}
C & \rightarrow D \\
A_1 & \rightarrow B_1 \\
A_2 & \rightarrow B_2
\end{align*}
\]

where \( A_1 \rightarrow B_1 \) and \( A_2 \rightarrow B_2 \) are the first two sides and \( C \rightarrow D \) is the third side. We represent this fact symbolically as

\[
(A_1 \rightarrow B_1) + (A_2 \rightarrow B_2) = (C \rightarrow D).
\]

Mathematics education is solidly grounded in psychology and philosophy among other fields (Schoenfeld, 2008, p. 467), and in a report (Skoleverket,
2003, p. 9–10) the names of three theories of learning used in Swedish schools are given to social constructivism, metacognitive theory, and symbolic interactionism. This explains the fact that Piaget and Vygotsky appear recurrently in the mathematics teacher education here in Sweden. In addition the names appear in a fashion of competing with each other. I believe if theories are used in a system of education then they should be consistent with each other. I thereby inquire into the question of what is the difference between the two giants. There certainly exists a great deal of differences, but when put them into the context of mathematics teacher education then the differences exist only at surface. The inquiry shows that, deep down, they are special cases of a united model. This model is the model for the cunning of reason which is the central point of the chapter 4. Under this model we find other models, for instance, the Marxist concept of labour.
Chapter 3

Three Examples

The three examples below are not examples in mathematics but in mathematics education, and the emphasis is not on the mathematical results but on the method by which the results are reached in an educational process. The method is the syllogistic analysis. In the examples the syllogistic analysis is carried out on some pieces of mathematics. The exact meaning of the syllogistic analysis together with that of the syllogism shall be given in the next chapter. The syllogistic analysis supplies us with interrelationships among pieces of mathematics grouped in certain way, and the information from the syllogistic analysis brings in educational order into the pieces.

3.1 Contingency and Necessity

This section is of auxiliary character and deals with a special syllogism generated by contingency and necessity. No syllogistic analysis shall be carried on it, what we want is merely a short discussion on some relationship between the contingent knowledge and the necessary knowledge by comparing it with a relationship between a stochastic variable and its samples in statistics. The discussion reveals the true meaning behind some terminology used in education.

When we take contingency as a thesis and necessity as a corresponding antithesis, then the resulting synthesis is the whole universe, for each event in the universe is either of contingency or else of necessity. The knowledge in contingency prevails everywhere, when you open your eyes you see a car, when you count you get a number. All these are contingent knowledge, for next time when you open your eyes you see something else, and when you count you get another number. The knowledge in necessity on the other hand is a production of thinking, thereby a piece of ultimate knowledge of science. In fact the most fundamental question within epistemology is of what is the necessary knowledge. This query has been investigated for thousands of years, yet satisfactory answers have not been reached, it is probably because
of the query is too big to be answered with just one stroke. However if we put the same query into some special disciplines then to certain extent some satisfactory answers can be reached. Smith (1993) reached some answers, and since his obtained results are not in the neighborhood of my concern, I shall as a result not go into it.

Instead of, I consider the pair of contingency and necessity in the context of mathematics education. Let us look at the following situation. When the teacher asks the student a question: “what is five?”, then student answers: “five is five books”. And some days later the teacher asks the same student the same question, then student answers: “five is five apples”. Some weeks later the teacher asks the same student the same question again, then student answers: “five is five balls”. With the above information what can the teacher judge on the necessary knowledge of the number five in the student? Can the teacher judge that the student has attained the necessary knowledge of the number five? If so, how can we explain the fact that the same student answers: “five is three apples” at the fourth time when the teacher asks the same question?

This very special example in fact represents a general phenomenon, that is to make a decision based on some finite empirical data. The most suitable decision, which is also in conformity with statistics, that the teacher can make is the following. The teacher judges that it is probable that the student has the necessary knowledge of the number five at the first answer of five is five books, or even better, the probability that the student has the necessary knowledge of the number five is 0.5. And the teacher judges that it is very probable that the student has the necessary knowledge of the number five at the second answer of five is five apples, or, the probability that the student has the necessary knowledge of the number five is 0.9. And the teacher judges that it is most likely that the student has the necessary knowledge of the number five at the third answer of five is five balls, or, the probability that the student has the necessary knowledge of the number five is 0.99. And at last the teacher judges that the probability that the student has the necessary knowledge of the number five is 0.95 and the probability that the student forgot it at that time is 0.88 at the fourth answer of five is three apples. We thereby judge the necessary knowledge of the student in terms of probability.

Because of contingency and necessity appear often in mathematics education, and because of it is important to know the terminology that is used therein, I shall for the sake of clarity discuss here some issues concerning a stochastic variable and its samples and refer a further discussion to, say, the book by Blom (1989).

Basic objects in probability as well as in statistics are stochastic variables sometimes even called random variables. Such variables are expressed by capital letters like $X, Y$. Let us assume that $X$ is the height of a random chosen person in a definite country. From the gathered human experiences
we can assume that this $X$ is ruled over by the normal law. When you for example visit that country and measure the first person you meet there, then you get say 182 cm, usually denoted by small letter $x$, in your case $x = 182$. The $x$ is a sample of size one of $X$ or an observation of $X$, and it is conveniently expressed as $X \rightarrow x$

\[
\text{stochastic variable } X \downarrow \text{observation or sample } x
\]

Let us look at the second example of stochastic variables. Assume that the student has taken a certain examination ten times under the condition that each time the student has 30% chance to get $G$ (passed) on the examination. Now let $X$ be the number of $G$ the student has among these ten tries. This variable is very different from the height variable considered above, for we can obtain by the mathematical reasoning the law that rules over $X$. The law is the binomial law.

There are all kinds of stochastic variables and each has its own law. Among all laws two are singularly important, the one is the normal law and the other is the binomial law. A principal task of statistics is to make decision on the law of a variable by making use of observations of the variable. The larger number of observations one can conduct the more precise decision one can make, but any finite number of observations, no matter how large, can not lead to the decision down to the last decimal. This consists in the law of large number in statistics.

Contingency in mathematics education often turns learning objects into some sort of stochastic variables. If say $A$ is a learning object, then the first thing concerns $A$ is that standard question of what is $A$, and the question is asked by necessity. Since it is necessary to ask a question, we can say that all learning objects are in the first place ruled over by the law of what. By contingency the answers can be of $A$ is $B$ or else of $A$ is $C$. For instance $A = \text{five}$, $B = \text{five books}$, $C = \text{five apples}$. It is therefore very natural to think of $B$ and $C$ as observations of $A$, hence symbols $A \rightarrow B$ and $A \rightarrow C$ become natural in which case the symbol $\rightarrow$ is interpreted as the word is. If we try to make this interpretation consistently throughout, then we can write the question of what is $A$ as $A \rightarrow \text{what}$. In this way we can write a typical event in mathematics education as

\[
A \quad A \quad A \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{what} \quad B \quad C \\
\quad \quad \quad \quad \ldots..\]

In the case where $A = \text{five}$, we have
The longer sequences are the better decisions on $A$ and five can be made in terms of probability. In what follows I shall adopt the conventional usage of the student has acquired the knowledge of the number five, however by this what I really mean is the probability that the student has acquired the knowledge of the number five is extremely high.

### 3.2 On $5 + 7 = 12$

The whole piece of this section is a giant syllogism, that is the synthesis of the ontological number and the operational number is the mathematical structure called group.

In the first part of this section, the ontological aspect of the number is discussed. I use the number five as my narrative example. Altogether six explicit syllogisms are presented and the syllogistic analysis is carried out on them. The syllogistic analysis reveals some interrelationships among them and shows that the last syllogism is advantageous in education for it carries implicitly with it a one-to-one correspondence relation generated by one’s own body.

In the second part of this section, the operational aspect of the number is discussed. The syllogistic analysis on three explicit syllogisms leads to the addition from the counting perspective. A further analysis leads to the subtraction as well.

The synthesis of the first part and the second part is the third part. I present it shortly for it stands at another level. My purpose here is merely to show the syllogistic way of progression.

### 3.2.1 The Concept of the Number Five

The identity of $5 + 7 = 12$ has five constituent parts; namely $5, 7, 12, +$ and $=;$ and the first three are of the same kind. If one has knowledge about say 5 it is then very unlikely that one does not have knowledge about 7 and 12, therefore I shall only discuss the number with the symbol 5. For the sake of writing I shall treat the text five and the symbol 5 indiscriminately.

The concept of the number five, as an educational task, starts with counting. Counting is counting material things in our universe. A material thing in our universe has three fundamental forms of existence. These three forms of existence are space, time and matter. The unification of space, time and matter is the motion; thus the most fundamental form of existence of a material thing is the motion.
A physical body of a human is made in such fashion that it can experience motion. Thus a human can distinguish things located at different places, things located at different time and things made of different materials. In order to experience a material thing it is not only necessary that the thing is being presented to me but also I must possess certain faculty of receiving that thing. This faculty is intuition.

Thus it is through intuition we begin with our experiencing material things. At the end of experiencing certain feeling, sense or knowledge are created in the mind carried by the experiencing body. The knowledge of the material thing in the mind is motionless, thereby is not that material thing which is in motion.

Inasmuch as knowing begins with experiencing through intuition, the statement of five is five books, emerging in the process of knowing the number five, must be taken as a random knowledge. Although random knowledge is not of scientific it has a great educational value, the statement of five is five books can be taken as a partial answer to the question of what is five. In order to proceed further I assume the sentence of five is five books is a synthetic by which I simply mean the sentence can not be further analyzed.

My inquiry into the number five consequently begins with a question of what is five. This question has two sides, they are the passive side and the active side. To be able to ask a question is not trivial, one has to learn to do it. The passive side of a question carries with it a passive taking, while the active side of a question supplies with an active searching. A moment of reflection leads to a partial answer of five is five books. Like the preceding question, it has the passive side and the active side as well. At the moment it is in its negative side. I shall group a question and a partial answer together as follows:

\[
\begin{align*}
\text{what is five?} \\
\text{five is five books}
\end{align*}
\]

which are the first two sides of a binary syllogism. It is called binary because two sides are prescribed. There is another kind of syllogism besides binary syllogisms, they are unitary syllogisms. I shall discuss much more about them in a later part of the essay.

A function of a syllogism is to produce the third side by combining the first two sides. Let us, regardless the grammar, write the question of five is what instead of the question of what is five. Then we have a more symmetric pair

\[
\begin{align*}
\text{five is what} \\
\text{five is five books}
\end{align*}
\]

Each of these two statements is a determinate denial of the other. Together they constitute a pair of thesis and antithesis which can also be called a dialectical pair.
According to one of the three general laws of Dialectics (Engels, 1940, p. 26) a thesis and its antithesis struggle to pass into each other and the struggle ends at a synthesis, this synthesis is the third side of the syllogism. As a result a statement of five is what becomes a premise of existence of a statement of five is five books and at the same time also an immediate construction of a statement of five is five books, and the other way around holds as well.

After a statement of five is what and that of five is five books have passed into each other, a new statement of five books is what is formed. What we saw here is a patter of progressive development

\[
\begin{align*}
\text{what is five?} & \quad \text{what is five books?} \\
\text{five is five books} &
\end{align*}
\]

Three sides have different functions. Generally, the third side is a result of act of thinking on the first side and the second side. Specially, the second side can be treated as a stimulus on the first side and the third side is a response. However there is something missing in this diagram which should represent the result of the first side and the second side have passed into each other. After the first side and the second side have passed into each other, not only the third side is produced by the first side and the second side, but also this process of producing is reversible. All these informations can be expressed by the following commutative diagram

\[
\begin{aligned}
five & \quad \xrightarrow{\text{II}} \quad \text{five books} \\
\text{I} & \quad \text{what} & \quad \text{III}
\end{aligned}
\]

where say II denotes the statement of five is five books, or what amounts to the same thing as five $\rightarrow$ five books. The commutativity means the synthesis of a question of what is five and a partial question of what is five books is an assertion of five is five books. Thereby the number five has been connected with the five-books via the mediation called what.

Let us write the fact that III is produced by the process of thinking on I and II as

\[I + II = III.\]

Then the reversibility is

\[II = III - I, \quad I = III - II,\]

and the meaning of these two equations is that I or II can be produced by II and III or I and III respectively. If the reversibility holds true in a syllogism then we say such a syllogism is stable.
This formulation of the reversibility is completely analogous to the reversibility used by Piaget (1985, p. 14) in his theory on equilibration of cognitive structures. As a result, I feel rather safe to say that the equilibration in Piaget is the stability in our context. For the moment, I shall push aside a discussion on Piaget, for I shall inquire into some of his works from the syllogistic perspective in a later chapter.

The discussion above happened to be associated with a synthetic statement of five is five books, for it was given to us by contingency. Because of contingency other synthetic statements of the same kind should be treated on an equal footing. Consequently the acquisition of the concept of the number five starts with all possible synthetic statements of five is five books, five is five horses, five is five apples, and so on. Each can be taken as a partial answer to the question of five is what, and thereby, precisely like we did above, leads to a new binary syllogism with the first two sides:

\[
\begin{align*}
\text{five is what} \\
\text{five is five apples}
\end{align*}
\]

A similar discussion leads to a synthesis of what is five-apples, and a commutative diagram

\[
\begin{array}{ccc}
five & \xrightarrow{\text{II}} & \text{five apples} \\
\downarrow{\text{I}} & & \downarrow{\text{III}} \\
\text{what} & &
\end{array}
\]

Thereby the query on the concept of the number five transforms to a query on all possible five material things. A commutative triangle is a syllogistic structure. The preceding two diagrams are two syllogistic structures.

Structure is not absolute, like anything else it has an antithesis. This antithesis is the de-structure. Likewise de-structure is not absolute either, and de-structure preconditions structure and paves the way for new structure. The development from structure to de-structure and then to re-structure is a unitary syllogism, for only one side, that is structure, is prescribed, the side de-structure is produced by the cunning of reason. We express it by the following commutative diagram:

\[
\begin{array}{ccc}
\text{structure} & \xrightarrow{R} & \text{re-structure} \\
\downarrow{T} & \downarrow{T^*} & \downarrow{\text{de-structure}}
\end{array}
\]
Many syllogisms used in mathematics and mathematics education are of this type, for instance the syllogism generated by the pair of general and special, and that of the pair of abstract and concrete, and so on. To see this we just write the process

\[ \text{abstract} \rightarrow \text{concrete} \rightarrow \text{abstract} \]

as

\[ \text{abstract} \rightarrow \text{de-abstract} \rightarrow \text{re-abstract}. \]

Now we go back the number five. After de-structure we have in front us many things, such as, five is five books, five is five apples, five is five rooms, and so on. We pick up one say, five is five books, and we know as well there are many similar things. In order to to see the difference we pick up another one say, five is five apples. Form a new syllogism starting with the pair of

\[
\begin{cases} 
\text{five is five books} \\
\text{five is five apples} 
\end{cases}
\]

In fact, it is more appropriate to view this process in the following fashion. Since the syllogisms

\[
\begin{cases} 
\text{five is what} \\
\text{five is five books} 
\end{cases}
\]

and

\[
\begin{cases} 
\text{five is what} \\
\text{five is five apples} 
\end{cases}
\]

are rather similar to each other, for they share the same first side; therefore they together generate a new syllogism:

\[
\begin{cases} 
\text{five is five books} \\
\text{five is five apples} 
\end{cases}
\]

In order to see the way that the statement of five is five books and that of five is five apples pass into each other, we construct a matrix

\[
\begin{pmatrix}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\circ & \circ & \circ & \circ & \circ
\end{pmatrix}
\]

in which \( \bullet \) means a book and \( \circ \) means an apple. Let us take a close look at this new pair and guess the way of passing into each other. Imagine there are five books on a desk in front of you and at the same time there are five apples in your hands. When five books pass into five apples and any pattern of arrangement of five books on that desk has an immediate construction on five apples. Of course the same holds the other way around.
Anyone with some mathematical knowledge can probably conclude that this way of passing into each other is the same thing as the one-to-one correspondence between the set of five books and the set of five apples. Namely we have a commutative diagram

![Diagram](https://via.placeholder.com/150)

where the generated third side III is the one-to-one correspondence relation between the set of five books and the set of five apples.

As we saw a one-to-one correspondence stems from a series of syllogistic structures, and a correspondence is a relation between two sets of things. Can we take a one-to-one correspondence as the concept of the number? This is a question at the foundation of mathematics, it does not concern us here. Nevertheless a one-to-one correspondence is clearly a good candidate for the concept of the number within mathematics education. If now I ask the student a question of what is the number five, and if the student answers that five is five books; then I would judge that the probability that the student has the knowledge of the number five is 0.2 If I ask the student the same question and he answers that five is five apples, then I would judge that the probability that the student has the knowledge of the number five is 0.7. If I ask the question of why I got two different answers for the same question, and if the student explains to me that the amount of books is the same as the amount of apples in terms of a one-to-one correspondence relation, directly or indirectly; then I would without any hesitation judge that the probability that the student has the knowledge of the number five is 0.999999999.

A one-to-one correspondence relation as the concept of number is a piece of dialectical knowledge, for it depends on a pair of sets. On the negative side it can be stated that the concept of five can not be attained by abstraction of just one set of books. Making one-to-one correspondences between two sets is organizing sets mathematically. This is an instance of mathematization which is a main goal of mathematics education. At least in acquisition of the concept of a number expressed in terms of a one-to-one correspondence between sets, a long process of mathematization is carried out by a series of syllogistic structures. A fine art of teaching and learning is to cut a long process into small pieces and to try to look upon those small pieces from the syllogistic perspective.

As I argued before, anyone with some mathematical knowledge can acquire the concept of five through mathematization. I am sure that the very same one, at early age, without any mathematical knowledge, used 5 and
counted by 5 confidently and fluently. I am also sure that each time when he used 5 and counted by 5, he used and counted in some contexts familiar to him. Those familiar contexts offer him stimuli to an inquiry into the number five. There are many stimuli available to him but the most important one comes from his own body. It is precisely in his own body there lies a dialectic upon number 5.

Any child uses fingers to point at things and uses simple language to talk about things. When a child counts, he points at a thing at each time or else he says of one, two, three, \ldots, or he points and talks at the same time. The child perceives the number of things through his arm or his tongue. Physiologically each time when one points or talks, a piece of information of muscular activity is registered and saved for a short while in one’s brain. The pair of the set of five books and the set of five muscular activities registered in the brain forms a canonical pair and the arm or the tongue gives rise to a canonical one-to-one correspondence relation. This is of course a unconscious mathematization which reaches into educational psychology as a principle of \textit{when one counts one is counted}. This principle is in fact a unitary syllogism, for it simply expresses the process:

\[
\text{one counts} \rightarrow \text{one is counted} \rightarrow \text{one knows how to count}.
\]

This important principle was already observed by Hegel.

The numbers produced by counting are counted in turn,

\[
\text{(Hegel, 2010, p. 172).}
\]

A child’s acquisition of concept of number at early age rests on his arms or tongue or other bodily organs and the consequential unconscious mathematization is more psychological and physiological than logical. At this age the child’s mathematical intuition is perception and sensation, and thereby a syllogism based on the pair of

\[
\begin{aligned}
\text{five} & \rightarrow \text{five books} \\
\text{five} & \rightarrow \text{five physical activities registered in the brain}
\end{aligned}
\]

plays an important role in mathematics education at early age and gives rise to the commutative diagram

\[
\begin{tikzpicture}
\node (A) at (0,0) {five};
\node (B) at (2,0) {five books};
\node (C) at (0,-1) {five physical activities};
\node (D) at (2,-1) {five physical activities};
\draw[->] (A) -- (B);
\draw[->] (A) -- (C);
\draw[->] (A) -- (D);
\end{tikzpicture}
\]
3.2. ON $5 + 7 = 12$

and the consequential matrix

$$\begin{bmatrix}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\circ & \circ & \circ & \circ & \circ
\end{bmatrix}$$

in which $\bullet$ means a book and $\circ$ means a physical activity. In plain language, the concept of the number at early stage comes from a relation of physical activities to material things. Together with the syllogistic principle that

one counts $\rightarrow$ one is counted $\rightarrow$ one knows how to count

it explain the reason the child can learn how to count without any systematic education.

3.2.2 Adding and Subtracting

As it was revealed in the last paragraph our didactical thoughts are built around interrelationships among certain syllogisms based on suitably chosen pairs of thesis and antithesis. We shall continue to do so in this paragraph.

We start with two syllogisms which we used in the preceding section. The one is

\[
\begin{align*}
\text{five is what} \\
\text{five is five books}
\end{align*}
\]

and the other is

\[
\begin{align*}
\text{seven is what} \\
\text{seven is seven books}
\end{align*}
\]

Two syllogisms such as these sometimes can be combined together to yield a new syllogism. The possibility depends on the forms of the syllogisms, for syllogism as structure is self-regulative. The self-regulation in structure is formulated by Piaget and shall be discussed in the next chapter. In our case the self-regulation simply means that the preceding syllogisms transform in to another syllogism if and only if the units are the same, namely book. If say the set of seven books is changed to the set of seven apples then the said syllogisms can not transform into another syllogism.

The simplest transformation is to put the syllogisms besides each other without modifying them, if they have the same forms. This leads to the addition from the counting perspective.

Adding means adding somethings to the existing things. The addition $5 + 7$ means to add 7 to 5 which can expressed syllogistically as

$$\begin{bmatrix}
5 \\
7 \\
12
\end{bmatrix}$$
This diagram involves three ideas which are fundamental in a mathematical discipline called Theory of Groups. The first idea is to treat a number, in our case 5 or 12, as an ontological object; the second is to treat a number, in our case 7, as an operational object; and the third is to treat an operational number, in our case 7 again, as a transformation or a motion from a number to another number, in our case the transformation from 5 to 12. In mathematics the addition is a binary operation which is a binary syllogism:

\[ c = a + b \]

and it is a commutative diagram.

The ontological five answers the question of what is five, while the operational five answers the question of what can five do. Therefore any number has two aspects, the ontological aspect and the operational aspect. A synthesis of the ontological aspect and the operational aspect is the transformational aspect. The syllogism of ontological, operational and transformational perspectives is the most fundamental perspective in Theory of Groups.

Since we discussed the ontological aspect of a number in the last paragraph, we shall therefore discuss in this paragraph the operational aspect and the transformational aspect.

The operational aspect of a number is counting. Counting starts with two given functionally different numbers. The one is an ontological number and the other is an operational number. In a process of counting, an ontological number is being treated as a center of counting, meaning a place where counting starts. When the center is the number zero, then there are two ways to count. The first way of counting is to say of one, two, three, \cdots while pointing at somethings. The second way of counting is to say of one, one, one, \cdots while pointing at somethings. We take counting two material things as an example. Those two ways of counting are united in

\[ 1 + 1 = 2 = 1 + 1. \]

A bonus of the process of counting as showed above is an educational definition of addition by one, that is \( a + 1 \). This makes any number rather than the number 1 less ontological, for a number is a sum \( 1 + 1 + \cdots + 1 \), for example

\[ 5 = 1 + 1 + 1 + 1 + 1, \]

so that there actually is only one ontological number, the number 1. I have to emphasize now that the statement that there is only one ontological number is valid only in mathematics education, not in mathematics as such.
The fact that any number is a sum of one’s makes a general addition rather easy. In the case of $5 + 7$, since $5 = 1 + 1 + 1 + 1 + 1$ and since

$$7 = 1 + 1 + 1 + 1 + 1 + 1 + 1,$$

then

$$5 + 7 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 12.$$ 

In an educational process, an attempt is made to simplify the addition. The simplicity is limited by the language used in the process. In all known natural languages the simplicity can be achieved by making larger number as a center of counting. In the case of $5 + 7$, it is more suitable to place the number 7 at a center of counting in which case counting is to say of eight, nine, ten, eleven, twelve. To find a suitable center in an addition is in fact important teaching point in elementary schools. For instance, in adding $401 + 6$, a center should be placed at 400, so that the addition is reduced to $1 + 6 = 7$. Since counting starts at 400, it follows that $401 + 6 = 407$.

After addition of two numbers is defined, then we turn our attention to the transformational aspect. The addition gives rise to a mathematical structure, an additive structure of non-negative integers. In this structure a sum $a + b$ of two integers $a$ and $b$ is defined in such a way that $a + b = b + a$, $a + (b + c) = (a + b) + c$ and $a + 0 = a$ hold true. Since the addition constructed above comes from counting, and counting in turn depends on the possibility of writing a number as a sum of 1’s; these additive properties hold automatically. In terms of the addition, a synthesis of the ontological aspect and the operational aspect is the transformational aspect. In mathematical terms each integer generates a transformation from the set of all non-negative integers into itself. If $b$ is a given integer, then the said transformation is

$$a + b = c$$

in which $b$ is an operational number while $a$ and $c$ are ontological numbers. The transformation means that a number $b$ transforms any number $a$ into another number $c = a + b$. If for instance $b = 5$ then 5 transforms 1 to 6, 12 to 17 etc. This kind of synthesis of the ontology and the operation exists not only in integers but also exists in any group. The so formed syllogism is the fundamental part of Representation Theory of Groups, which plays an extremely important role in Quantum Theory.

In order to proceed further let us examine the additive operation more closely. In the preceding syllogism concerning $a + b = c$, let us take a look at a simple case, the case where $b = 1$. The addition $a + 1 = c$ is counting by one at the center $a$. The said syllogism in the simplified case is
Now we impose the stability on the diagram and this means that one is added to the center \(a\) to yield \(a + 1\) and at same time \(a\) can be recovered from \(c\). The latter certainly turns \(c\) into a center of counting. Now the situation is clear, to recover \(a\) from \(c\) is to take away one from \(c\). That is

\[c - 1 = a.\]

This is the subtraction by one. In the same fashion to recover \(a\) from \(a + b = c\) is to subtract \(b\) from \(c\), that is \(c - b = a\). Together we have

\[a + b - b = a\]

which means the operation of adding by \(b\) is canceled by subtracting by \(b\), in symbols,

\[b - b = 0.\]

With counting; 5 + 7 = 12, 2 + 2 = 4, 1 + 1 = 2 are trivial. Let us see what happens if we take counting away. Both Leibniz and Poincaré proved 2 + 2 = 4 without using counting but at cost of heavy assumptions.

In Leibniz’ proof presented by Frege (1953, p. 7), the assumptions are 1 + 1 = 2, 2 + 1 = 3, 3 + 1 = 4 and \(a + (b + c) = (a + b) + c\); and the proof runs as

\[2 + 2 = 2 + (1 + 1) = (2 + 1) + 1 = 3 + 1 = 4.\]

Poincaré (1913, pp. 32–33), on the other side, assumes 1+1 = 2, 2+1 = 3, 3 + 1 = 4 and \(x + 2 = (x + 1) + 1\); and argues

\[2 + 2 = (2 + 1) + 1 = 3 + 1 = 4.\]

A similar proof of 1 + 1 = 2 is also provided by Ernest (1991, p. 5). The assumptions are \(s0 = 1\) (meaning 0 + 1 = 1), \(s1 = 2\) (meaning 1 + 1 = 2), \(x + 0 = x\) and \(x + sy = s(x + y)\) (meaning \(x + (y + 1) = (x + y) + 1\)); and the proof runs as

\[1 + 1 = 1 + s0 = s(1 + 0) = s1 = 2.\]

Counting resigns itself to intuition which manifests in seeing, perceiving and sensing; but intuition alone can not supply us with anything if it is not permitted to act on the material world. Thus, in addition to seeing, perceiving and sensing treated as inner parameters to intuition; there exist as well
outer parameters. Light and gravitation are definitely sources of these outer parameters. Formalization of intuitive counting becomes formalized counting which results in arithmetic. Formalization in terms of inner parameters leads to numbers. Our seeing, perceiving and sensing have character of being clear and distinct; and our sense of numbers consequently is psychologically safe and seldom in dispute.

3.2.3 The Additive Group of Integers

The system of all numbers constitutes a universe of numbers in which there exists a transitive motion represented by the addition. Any two numbers can be moved to each other at will by a motion, and motion is a unity or duality of the ontological and operational aspects of numbers. The law of conservation, that is \(a - a = 0\), transforms the universe into a additive group. A additive group is a set \(G\) together with a commutative binary operation written as \(a + b\). The commutativity means \(a + b = b + a\). The operation satisfies four conditions listed below:

1. \(a + b\) is in \(G\) if \(a, b\) are in \(G\),
2. \((a + b) + c = a + (b + c)\) if \(a, b, c\) are in \(G\),
3. there exists 0 in \(G\) such that \(a + 0 = 0 + a = a\),
4. for each \(a\) there is a \(-a\) such that \(a + (-a) = 0\).

3.3 On 1/2 + 1/3

I believe many mathematics educators ask the following question: what is the student thinking about when the student writes down

\[
\frac{1}{2} + \frac{1}{3} = \frac{2}{5}?
\]

and what are causes for that?. I would say the student is thinking about counting and one of causes is habit. By the time the student begins the study of fractions, the student has already mastered counting, for the student knows \(1 + 1 = 2\) and \(2 + 3 = 5\). In fact the student masters counting so well that a habit has rooted inside the student. The habit causes the student see numbers of all kinds from the counting perspective. A task of mathematics educators is to help the student break that habit and change the perspective.

In order to help the student to leave the counting perspective, numerical values of fractions should be avoided at maximum, instead, attention should focused on the forms of fractions. Forms of fractions appear at random and depend on two integers, thereby manipulation of fractions is entirely different from that of integers, though the mathematical laws governing integers and fractions are the same. Curriculum material on fractions is rich and is still growing. Most of the curriculum material is about helping the student
in adding and comparing two fractions and one recent work is (Hill and Charalambous, 2012).

The above mentioned incorrect addition shows clearly that counting itself is not sufficient for dealing with fractions, so that we must examine counting from its negative side, this is de-counting. That looking at the negative side of counting does not mean counting is abandoned altogether. The aim of de-counting is to sublate counting itself, and this sublated counting is re-counting. In the present case re-counting is measuring. The progression

\[ \text{counting} \rightarrow \text{de-counting} \rightarrow \text{re-counting} \]

is another unitary syllogism.

The first part of this section is about the numerical aspect of fractions, it is merely pointed out that operations on fractions based on their numerical values differ insignificantly from those of integers.

The second part of the section is a syllogistic analysis on three syllogisms. The analysis gives rise to a graphical way of seeing fractions.

All discussion here and from the last section leads to the third part. Like the third part in the last section, a mathematical structure called field is an advanced topic which will not be picked up here.

### 3.3.1 Numerical Values of Fractions

The number 0.5 is a common numerical value of the fractions 1/2, 2/4, 23/46. In fact every fraction has a unique numerical value. Some numerical values are expressed by numbers with finitely many digits while others are not to which belongs the fraction 1/3. In the perspective of numerical value the addition of fractions differs insignificantly from the addition of integers. For instance the numerical value of 1/2 is 0.5 and that of 3/5 is 0.6 then it follows that

\[ \frac{1}{2} + \frac{3}{5} = 0.5 + 0.6 = 1.1 \]

which together with the fact that the numerical value of the fraction 11/10 is 1.1 gives rise to

\[ \frac{1}{2} + \frac{3}{5} = \frac{11}{10}. \]

The difficulty with fractions lies in the inference of relations among fractions without referring to their numerical values. In the example given above this means we have to infer a relation 1/2+3/5 = 11/10 without appealing to 0.5, 0.6 and 1.1. An educational challenge is therefore a battle of a seemingly reasonable but incorrect inference 1/2 + 3/5 = 4/7 against a chaotically unreasonable but correct inference 1/2 + 3/5 = 11/10.
3.3.2 Fractions

Before dealing with fractions seriously, it is important to examine counting from the perspective of

counting → de-counting → re-counting.

This and

\[
\begin{cases}
\text{five is what} \\
\text{five is the result of counting five books}
\end{cases}
\]

will be the syllogisms on which a syllogistic analysis shall be executed. The analysis shall ease a transition from counting perspective to measuring perspective. It is the latter that really matters in dealing with fractions.

Let us begin with a new examination of an old counting situation once more. We would like to examine the way we count the bullets • • • in front of us. An old way of counting is to point at the bullet to the left while uttering one, and then move the finger one step to the right while saying two, and then move the finger further to the right while saying three. By this counting we may conclude the number of bullets in front of us is 3. Do we have another perspective of seeing this procedure? Searching for another perspective of counting amounts to de-counting. Counting appears in different forms. Counting of bullets we just did is counting on discrete objects. In counting the length of a segment $CD$ by another segment $AB$ as showed in the following figure:

\[
\begin{array}{c}
\text{C} \\
A
\end{array}
\begin{array}{c}
\text{D} \\
B
\end{array}
\]

if the segment $AB$ is one meter then we conclude the length of $CD$ is three meters. This counting is different from counting of bullets made above in that the former uses the finger (a ruler inside us) as counting unit while the latter uses the segment $AB$ (a ruler outside us) as counting unit. Thereby counting bullets is a subjective counting while counting length is an objective counting. Object counting is measuring. The unitary syllogism

\[
\text{counting} \rightarrow \text{de-counting} \rightarrow \text{re-counting}
\]

consequently becomes

\[
\text{subjective counting} \rightarrow \text{de-subjective counting} \rightarrow \text{measuring}
\]

Said so, now we reexamine counting from the measuring perspective. Measuring is comparing two things of the same kind, and in a process of measuring one thing (to measure) is active and another (to be measured) is
passive. In conformity with the figure of measuring $CD$ by $AB$ made above, we now, in counting the bullets •••, create the following measuring matrix

$$
\begin{pmatrix}
\bullet & \bullet & \bullet \\
\bullet & & \\
\end{pmatrix}
$$

It is used in the same way as that in the figure of measuring $CD$ by $AB$, meaning to use the second row to measure the first row. The first row has 3 bullets and the second row has 1 bullet, and the result of measuring is 3. There is in fact no better way than

$$
\frac{3}{1} = 3
$$

to record this measuring process. The shift from 3 to $3/1$ is a shift from counting to measuring, it is a big shift, but it is taken almost as granted in any educational process. Good use of the shift can prevent the student making the mistake described at the beginning of this paragraph. In order to see this, let us reexamine the addition of two numbers, say $3 + 4$.

The number 3 is

$$
3 = \frac{3}{1}
$$

with its measuring matrix

$$
\begin{pmatrix}
\bullet & \bullet & \bullet \\
\bullet & & \\
\end{pmatrix}
$$

and the number 2 is

$$
2 = \frac{2}{1}
$$

with its measuring matrix

$$
\begin{pmatrix}
\bullet & \bullet \\
\bullet & \\
\end{pmatrix}
$$

and to measure $3 + 2$ we have another measuring matrix

$$
\begin{pmatrix}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & & & & \\
\end{pmatrix}
$$

The last measuring matrix gives rise to

$$
\frac{5}{1} = 5.
$$

Therefore

$$
\frac{3}{1} + \frac{2}{1} = \frac{5}{1}
$$

not

$$
\frac{3}{1} + \frac{2}{1} = \frac{5}{2}.
$$
It is very suitable to record what been done coherently as

\[
\left( \begin{array}{c}
\circ \\
\circ \\
\circ 
\end{array} \right) + \left( \begin{array}{c}
\circ \\
\circ \\
\circ 
\end{array} \right) = \left( \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ 
\end{array} \right)
\]

and call it *principle of the same ruler*. Since a function of the second row is a ruler, thereby the principle states the objects measured are added but the ruler is kept the same. The principle supplies with us the first example of showing what you should do and what you should avoid when you add two fractions. The change of perspective from counting to measuring not only sublates natural numbers but also paves the way for fractions.

It is the syllogistic analysis that led us to measuring, and now it is also this analysis which shows us a path of progression. Now we introduce a unitary syllogism:

when one measures one is measured.

Let us take the measuring matrix for 3 as our narrative example:

\[
\left( \begin{array}{c}
\bullet \\
\bullet \\
\bullet 
\end{array} \right)
\]

In the measuring situation made above, we measure \( \bullet \bullet \bullet \), the whole, by \( \bullet \), the part. But the newly mentioned syllogism implies this is only half of a story, it is in fact completely legitimate to measure the part by the whole, that is to measure \( \bullet \) by \( \bullet \bullet \bullet \). This motivates us to look at another measuring matrix

\[
\left( \begin{array}{c}
\bullet \\
\bullet \\
\bullet 
\end{array} \right)
\]

The result of measuring is

\[
\frac{1}{3}.
\]

The ontological \( 1/3 \) presents a great deal of educational difficulties, and many curriculum materials have been designed to offer \( 1/3 \) an educational context, the Pizza model is one of them. I shall discuss another model in a later chapter. However the operational \( 1/3 \) is much easy to handle, but it demands certain ability of using symbols. The operational \( 1/3 \) is based on the consideration that whichever \( 1/3 \) can be it must be such a number that

\[
\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1
\]

or what amounts to the same thing as

\[
3 \cdot \left( \frac{1}{3} \right) = 1
\]
which can be seen from
\[
\begin{align*}
\begin{pmatrix}
\cdot & \cdot & \cdot \\
\end{pmatrix}
+ \begin{pmatrix}
\cdot & \cdot & \cdot \\
\end{pmatrix}
+ \begin{pmatrix}
\cdot & \cdot & \cdot \\
\end{pmatrix}
= \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}
\end{align*}
\]
This is another instance on the principle of the same ruler. A natural generalization of the principle of the same ruler now is easy to make. In adding two fractions, the first step is to write down their measuring matrices, and then to check the second rows; if the second rows contain equally many objects then adding the fractions amounts to adding the first rows. This is the general principle of the same ruler.

The general principle of the same ruler is not sufficient for adding two fractions with different denominators. In order to do that, a principle of cloning measuring is necessary. Let us take \( \frac{2}{3} \) as our narrative example. Cloning the measuring matrix for \( \frac{2}{3} \), so that we have a number of clones of the matrix, say three clones:

\[
\begin{align*}
\begin{pmatrix}
\cdot & \cdot & \cdot \\
\end{pmatrix},
\begin{pmatrix}
\cdot & \cdot & \cdot \\
\end{pmatrix},
\begin{pmatrix}
\cdot & \cdot & \cdot \\
\end{pmatrix}
\end{align*}
\]

The principle of cloning measuring says that if all these clones are jointed together to make a new measuring matrix, in the sense that all the first rows are jointed together and all the second rows are jointed together, then the measuring result of the newly formed measuring matrix is the same as the measuring result of only one measuring matrix without being jointed by others. This means if we define

\[
\begin{align*}
\begin{pmatrix}
\cdot & \cdot & \cdot \\
\end{pmatrix} \cup \begin{pmatrix}
\cdot & \cdot & \cdot \\
\end{pmatrix} \cup \begin{pmatrix}
\cdot & \cdot & \cdot \\
\end{pmatrix}
= \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}
\end{align*}
\]

then

\[
\begin{align*}
\begin{pmatrix}
\cdot & \cdot & \cdot \\
\end{pmatrix} \cup \begin{pmatrix}
\cdot & \cdot & \cdot \\
\end{pmatrix} \cup \begin{pmatrix}
\cdot & \cdot & \cdot \\
\end{pmatrix} \cup \begin{pmatrix}
\cdot & \cdot & \cdot \\
\end{pmatrix}
= \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}
\end{align*}
\]

Mathematically this is

\[
\frac{2 \cdot 3}{3 \cdot 3} = \frac{2}{3}.
\]

With these two principles we can add two fractions at will. Let us add \( \frac{1}{2} + \frac{2}{3} \). By the principle of cloning measuring, the measuring matrix for \( \frac{1}{2} \) is

\[
\begin{pmatrix}
\cdot & \cdot \\
\end{pmatrix} = \begin{pmatrix}
\cdot & \cdot \\
\end{pmatrix} \cup \begin{pmatrix}
\cdot & \cdot \\
\end{pmatrix} \cup \begin{pmatrix}
\cdot & \cdot \\
\end{pmatrix} = \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}
\]
3.4. ON THE EQUATION $AX^2 + BX + C = 0$

and

\[
\begin{pmatrix} \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{pmatrix} \cup \begin{pmatrix} \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}
\]

Now it follows from the principle of the same ruler that

\[
\begin{pmatrix} \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{pmatrix} + \begin{pmatrix} \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{pmatrix} + \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}
\]

\[
= \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}
\]

This shows that

\[
\frac{1}{2} + \frac{2}{3} = \frac{7}{6}.
\]

3.3.3 The Field of Rationals

A number of the form $\frac{p}{q}$ is a rational number. The system of all rational numbers constitutes a new universe, a better universe than that of all integers. Thus, in this universe, not only the addition and the subtraction can be carried out freely, but also the multiplication and the division. The addition and the multiplication transform the system of all rational numbers into a field. A field is a set $F$ together with two commutative binary operation written as $a + b$ and $a \cdot b$. With respect to the addition $+$, the set $F$ is an additive group; with respect to the multiplication $\cdot$, all non-zero elements in $F$ is a multiplicative group.

3.4 On the Equation $ax^2 + bx + c = 0$

The equation in the title is a synthesis of $a, b, c$ on the one hand and $x, y, z$ on the other. Letters $a, b, c$ in mathematics represent constants while $x, y, z$ represent variables. Constants are unchanging whereas variables are changing, thereby in the broadest sense an equation is a special case of syllogism generated by changing and unchanging:

```
world
/   \
changing     unchanging
```

The first part of this section conducts a short discussion on a mathematical concept called constant, while the second part is about a mathematical concept called variable.
The third part of this section conducts a syllogistic analysis on the equation in the title. Inasmuch as equations are syllogisms, an emphasis of syllogistic analysis on equations is on interrelationships among equations not on equations themselves. To solve an equation say

\[ f(a, b, c; x, y, z) = 0 \]

means to relate it through a series of intermediate equations to equations

\[ x = g_1(a, b, c), \quad y = g_2(a, b, c), \quad z = g_3(a, b, c) \]

which is called a solution of the equation \( f(a, b, c; x, y, z) = 0 \).

### 3.4.1 Constants

The mathematical concept of constant is not a completely trivial one. Firstly, constants are almost always denoted by \( a, b, c \), etc, the beginning letters in the alphabet; whereas variables are denoted by \( x, y, z \), etc, letters in the end of the alphabet. A possible confusion arising from constants is their relativeness, thereby a solid understanding of constants depends on the background as well. This means when you look at some constants you have to look at the variables in the background.

Secondly, another possible confusion comes from an identification of constants with known quantities in an educational process; for a constant can be unknown while a variable can be known. The number \( \pi \) is the constant but unknown in the numerical sense; while the season of a year is a variable but known.

The confusions can be avoided by simply looking at constants and variables at the same time and in the same context.

### 3.4.2 Variables

Much of the text written above on constants is also valid for variables, if the word *constant* is replaced by *variable* in that text, except for the second part.

An identification of variables with unknown quantities presents much unnecessary confusion, for it leads automatically to a question of if \( x \) is unknown then how I am going to find it. Thereby, if the word *unknown* is being used then there must be further explanation about the word; for instance, to use *temporarily unknown* in place of *unknown*.

### 3.4.3 Equations

As we argued above, an equation is a synthesis of constants and variables, so that an equation, as a structure, is a syllogism. When constants are \( a, b, c \)
and variable is $x$, an equation is said to be an equation on $x$ and determined by $a, b, c$. Such an equation is not necessarily dependent on $x$, for instance

$$a + b - c = 0$$

in which case the equation is a constraint on the constants. We can classify equations on $x$ and determined by $a, b, c$ by making use of the degree. An equation of the first degree on $x$ and determined by $a, b, c$ is

$$ax + b = c,$$

and an equation of the second degree on $x$ and determined by $a, b, c$ is

$$ax^2 + bx + c = 0.$$

The meaning of solving an equation $ax^2 + bx + c = 0$ mathematically is to search for another equation of type $x = f(a, b, c)$ and to carry out a syllogistic analysis between them. The emphasis of a syllogistic analysis is on interrelationships among equations not on equations themselves. As a typical example of such an analysis, we write down three equations:

$$a = b, \quad a + c = b + c, \quad a \cdot c = b \cdot c.$$

The syllogistic analysis states that the equation $a = b$ and the equation $a + c = b + c$ transform into each other, and that the equation $a = b$ and the equation $a \cdot c = b \cdot c$ also transform into each other. In symbols,

$$a = b \iff a + c = b + c$$

and

$$a = b \iff a \cdot c = b \cdot c.$$

Let us call the first additive transformation law and the second multiplicative transformation law in a syllogistic analysis.

In this spirit let us conduct a syllogistic analysis on the equation $ax^2 + bx + c = 0$.

When dealing with a complex thing it is always beneficial to begin with simple things. An equation such as

$$ax^2 + bx + c = 0$$

has three general coefficients $a, b, c$ which include three special cases:

$$0, b, c; \quad a, 0, c; \quad a, b, 0.$$

In the first special case where $a = 0$, our equation becomes

$$bx + c = 0.$$
A solution of the equation can be constructed easily by a syllogistic analysis. By the additive transformation law, the equation $bx + c = 0$ and the equation

$$(bx + c) + (-c) = 0 + (-c)$$

transform into each other, and the second equation is

$$bx = -c.$$  

Now by the multiplicative transformation law, the equation $bx = -c$ and the equation

$$xb \cdot \left(\frac{1}{b}\right) = -c \cdot \left(\frac{1}{b}\right)$$

transform into each other, and the last equation is

$$x = \frac{-c}{b}.$$  

The syllogistic analysis thereby concludes the equation $bx + c = 0$ and the equation

$$x = \frac{-c}{b}$$

transform into each other, in other words, $x = -c/b$ is a solution of the equation $bx + c = 0$.

In the second special case where $b = 0$, our equation $ax^2 + bx + c = 0$ becomes

$$ax^2 + c = 0.$$  

In order to solve this equation, besides the additive transformation law and the multiplicative transformation law, we need a square transformation law as well. This law states that the equation $x^2 = d$ and the equation $x = \pm \sqrt{d}$ transform into each other, in symbols,

$$x^2 = d \iff x = \pm \sqrt{d}.$$  

By the almost exact same procedure as that in the first case, the syllogistic analysis concludes the equation $ax^2 + c = 0$ and the equation $x = \pm \sqrt{-c/a}$ transform into each other, or a solution to the equation $ax^2 + c = 0$ is

$$x = \pm \sqrt{-\frac{c}{a}}.$$  

In the third special case where $c = 0$, our equation $ax^2 + bx + c = 0$ becomes

$$ax^2 + bx = 0$$

which can be transformed into $x(ax + b) = 0$. From the last equation we have either $x = 0$ or $ax + b = 0$. Solving the equation $ax + b = 0$ as above, we obtain either $x = 0$ or $x = -b/a$. 

After these three special cases have been treated, if we now reflect over what we have done, then we would notice that besides the three case, another case can be treated as well. This new case is an equation \( a(x + d)^2 + c = 0 \), for a solution is
\[
x = -d \pm \sqrt{-\frac{c}{a}}.
\]

The equation \( a(x + d)^2 + c = 0 \) and the equation \( ax^2 + bx + c = 0 \) are so similar that they together lead to a question of how transform they into each other.

Let us put them together
\[
\begin{cases}
ax^2 + bx + c = 0 \\
a(x + d)^2 + c = 0
\end{cases}
\]
and then observe their forms. They have other forms as well:
\[
\begin{cases}
ax^2 + bx + c = 0 \\
ax^2 + 2adx + ad^2 + c = 0.
\end{cases}
\]

We change the forms a bit
\[
\begin{cases}
ax^2 + 2a(b/2a)x + a(b/2a)^2 - a(b/2a)^2 + c = 0 \\
ax^2 + 2adx + ad^2 + c = 0.
\end{cases}
\]

This motivates the following transformation matrix
\[
\begin{pmatrix}
d & c \\
\downarrow & \downarrow \\
(b/2a) & -a(b/2a)^2 + c
\end{pmatrix}
\]
By this matrix, the equation \( a(x + d)^2 + c = 0 \) and the equation \( ax^2 + bx + c = 0 \) transform into each other. Since a solution to the equation \( a(x + d)^2 + c = 0 \) is
\[
x = -d \pm \sqrt{-\frac{c}{a}},
\]
by the transformation matrix a solution to the equation \( ax^2 + bx + c = 0 \) is
\[
x = -\frac{b}{2a} \pm \sqrt{\frac{a(b/2a)^2 - c}{a}} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.
\]

This completes the searching for a solution of the equation \( ax^2 + bx + c = 0 \).

In the case where \( a = 1 \), \( b = p \) and \( c = q \), the equation \( ax^2 + bx + c = 0 \) becomes
\[
x^2 + px + q = 0.
\]
A solution becomes 
\[ x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}. \]

The last formula is often called the \((p, q)\)-formula for the quadratic equation \(x^2 + px + q = 0\).

The rest of this section is about some comments on teaching of the quadratic equation \(x^2 + px + q = 0\).

This equation has an interesting place both in mathematics and mathematics education. In the latter, one of learning difficulties occurs repeatedly, and it manifests itself in computation of the solutions based on the \((p, q)\)-formula. Actually, the difficulty is not in the computation itself but rather in the stage of preparation. For instance, in solving the equations
\[
\begin{align*}
x^2 + 5x + 6 &= 0, \\
x^2 - 9x + 20 &= 0, \\
x^2 - x - 12 &= 0
\end{align*}
\]
it is quite common to pick \(p\) and \(q\) as 5 and 6 in the first equation, as 9 and 20 in the second, as \(\pm 1\) and 12 in the third. Of course, the difficulty can be reduced substantially by emphasizing the way of writing the equations, namely, the second equation and the third equation should be written as
\[
\begin{align*}
x^2 + (-9)x + 20 &= 0, \\
x^2 + (-1)x + (-12) &= 0.
\end{align*}
\]

In this way, the transformation matrix
\[
\begin{pmatrix}
p & q \\
\downarrow & \downarrow \\
-9 & 20
\end{pmatrix}
\]
makes easy to see the way that the equation \(x^2 + px + q = 0\) and the equation \(x^2 + (-9)x + 20 = 0\) transform into each other, and the transformation matrix
\[
\begin{pmatrix}
p & q \\
\downarrow & \downarrow \\
-1 & -12
\end{pmatrix}
\]
makes easy to see the way that the equation \(x^2 + px + q = 0\) and the equation \(x^2 + (-1)x + (-12) = 0\) transform into each other. I end the discussion by mentioning an empirical study on the difficulty conducted by Olteanu (2007).

Modern teaching on the quadratic equation \(x^2 + px + q = 0\) emphasizes too much on the \((p, q)\)-formula and ignores others. Sometimes it is precisely other methods that are really interesting from the educational perspective. I shall exemplify it by the Viète’s formula.

Assume that \(r_1\) and \(r_2\) are two solutions to the quadratic equation. Then we have
\[ x^2 + px + q = (x - r_1)(x - r_2). \]
If we develop the right side then we get

\[ x^2 + px + q = x^2 - (r_1 + r_2)x + r_1r_2 \]

so that

\[
\begin{cases}
  r_1 + r_2 = -p \\
  r_1 \cdot r_2 = q
\end{cases}
\]

which is the Viète’s formula. The formula offers a simple way of solving the quadratic equation. Given a quadratic equation \( x^2 + px + q = 0 \). If you can factorize \( q \) into \( r_1 \cdot r_2 \) and to add the factors together and to make sure the result of addition is \(-p\), then you can also factorize the quadratic polynomial \( x^2 + px + q \), namely,

\[ x^2 + px + q = (x - r_1)(x - r_2). \]

Hence the equation \( x^2 + px + q = 0 \) becomes \((x - r_1)(x - r_2) = 0\) whose solution is trivially \( x = r_1 \) or \( x = r_2 \).

Some examples are in order. For the polynomial \( x^2 + 5x + 6 \), since \( 6 = (-2) \cdot (-3) \) and \(-2 + (-3) = -5\), it follows that

\[ x^2 + 5x + 6 = (x + 2)(x + 3). \]

The solution of \( x^2 + 5x + 6 = 0 \) is \( x = -2 \) or \( x = -3 \). For the polynomial \( x^2 - 9x + 20 \), since \( 20 = 4 \cdot 5 \) and \( 4 + 5 = 9 \), it follows that

\[ x^2 - 9x + 20 = (x - 4)(x - 5). \]

The solution of \( x^2 - 9x + 20 = 0 \) is \( x = 4 \) or \( x = 5 \). For the polynomial \( x^2 - x - 12 \), since \(-12 = (-3) \cdot 4 \) and \(-3 + 4 = 1\), it follows that

\[ x^2 - x - 12 = (x + 3)(x - 4). \]

The solution of \( x^2 - x - 12 = 0 \) is \( x = -3 \) or \( x = 4 \).

I used the syllogistic analysis to solve the quadratic equation which can also be solved by the method of completing square. The foundation of the latter is a formula

\[ (A + B)^2 = A^2 + 2AB + B^2. \]

The formula has two close relatives

\[ (A - B)^2 = A^2 - 2AB + B^2, \quad (A + B)(A - B) = A^2 - B^2. \]

The standard way to deduce these formulas is to use the distributivity, in sense that these formulas are the didactic transpositions of the distributivity. But I shall not do that here, for I did it many times in my teaching. I know that the standard way is logically safe but not didactically motivated, for the student could not see the reason behind the deduction. It seems that
the modern curriculum and textbooks emphasize the formal part of the rule and pay less attention to the intuitive part. Intuition often relates itself to geometric figures, a proper figure enhances a great deal understanding on the formal deduction, thereby should be taught side by side formal rules.

Consider a syllogism generated by the quadratic rule

\[(A + B)^2 = A^2 + 2AB + B^2\]

and the geometric figure

\[
\begin{array}{ccc}
B & AB & B^2 \\
A & A^2 & AB \\
A & B
\end{array}
\]

because the area of that large quadrate is \((A + B)^2\) which is equal to the sum of \(AB, A^2, AB\) and \(AB\); the sum is clearly \(A^2 + 2AB + B^2\).

Consider also a syllogism generated by

\[(A - B)^2 = A^2 - 2AB + B^2\]

and the geometric figure

\[
\begin{array}{ccc}
A & (A - B)B & B^2 \\
B & (A - B)^2 & (A - B)B \\
A - B & (A - B)^2 & (A - B)B \\
A - B & B
\end{array}
\]
because the area of the large quadrate is $A^2$ which is the sum of $2(A - B)B$, $(A - B)^2$ and $B^2$, namely,


Consider at last a syllogism generated by the formal rule

$$(A + B)(A - B) = A^2 - B^2$$

and the geometric figure

because the area of the large rectangle is $A(A + B)$ which is equal to the sum of $AB$, $B^2$ and $(A + B)(A - B)$, namely,

$$A^2 + AB = AB + B^2 + (A + B)(A - B)$$

which is


I firmly believe these three syllogisms improve substantially a true understanding of the quadratic rule and its relative.

A word of warning is in order. Using geometric figures in this way is not positioning a metacognitive shift, for the emphasis is on both algebraic manipulation and geometric intuition. A geometric figure and its corresponding algebraic equation are didactic transpositions of the same piece of mathematical knowledge. In the case of geometry and algebra, they should be shifted constantly. Using only geometric figures is a metacognitive shift, which may lead the student’s mathematics to take a quite different form from that of the mathematician (Kang and Kilpatrick, 1992).

The duality between geometric figures and algebraic equations is embedded in the much larger duality between body and mind. This duality was initiated by Descartes as the method in searching for knowledge (see Chapter 4). The knowledge obtained in this way has character of being clear and distinct. As an application of this theory, Descartes created the so-called Analytic Geometry. The fundamental method in this geometry is to relate
a geometric figure with an algebraic equation via the duality between body and mind. In the theory of didactic transposition, this means knowledge has two equally valuable didactic transpositions, one is geometric and other is algebraic.
Chapter 4

The Syllogistic Analysis

In order to describe what the examples in the last chapter indicates and what common trait the examples have, we need first of all to search for a framework. Syllogism, transformation, self-regulation and interrelationship, these are the words used in the last chapter, and they indicate that the searching should begin with structuralism.

The first section is thereby about structuralism. I begin with the formulation given by Piaget and Cherryholmes, and then I modify the formulation and record it in the spirit of axiomatics. In this axiomatic definition of structuralism, a meaning of the structural analysis becomes clear and distinct. It turns out that harmonic analysis in mathematics is a special case of structural analysis. If a structure in structuralism is interpreted as a syllogism then the structural analysis is the syllogistic analysis. With this framework, a common trait of the examples in the last chapter is the syllogistic analysis which were carried out on them.

The second section is about the philosophy on mathematics held by Bourbaki. The purpose of this section is to show the syllogistic analysis lies at the very bottom of mathematics in Bourbaki.

The third section is devoted to a discussion on the syllogism itself, for after all it is one of basic objects of the essay.

The fourth section conducts a discussion on form of sides in a syllogism, in particular the form of sides appearing in mathematics education. The search for a general form ends at the form of $A$ is $B$, and the section ends with an important concept of stability in syllogisms.

4.1 Structural Analysis

The key word in the title is *structural*. The meaning of this word is intimately related to that of structuralism. Thereby I begin with the search for a meaning of structuralism.

What is structuralism in the broadest sense? I asked this question myself
many times, but never reached a satisfactory answer which can be expressed by just a few words. No matter what structuralism should be; at the very bottom, it concerns itself certain way of thinking or doing on the philosophical plane. In any try of defining structuralism, language is being used, and you have to define structuralism by language. You have to define it orderly not chaotically, that demands you have to define structuralism structurally. Now it looks like a situation in mathematics where mathematicians deal with the set of all sets. If that is the case then the search for the complete meaning of the word structuralism would lead to very bottom of philosophy. That is not the aim of this essay, and thereby I shall not attempt to determine the absolute meaning of structuralism, instead, I shall first look at the meaning given by others, and then try to modify it suitably in such a fashion that I can write down a relative meaning of the word with a hope that this relative meaning can express clearly and distinctly the essence of the essay.

There is a very good reason that I start with Piaget, for it was him who formulated the three laws for structuralism. He is said to be one of the creators of constructivism, therefore he is in one way or other involved in structuralism. This is indeed the case. Piaget (1971) made an attempt to define structuralism and the description he gave takes 16 pages. I am rather sure that after you have read these 16 pages you still ask a question as I asked at the beginning of the paragraph. However a reading of these pages is not completely in vain, for Piaget singled out three important aspects of structuralism in extremely abstract form. The aspects are wholeness, transformation and self-regulation. In order to illustrate the meaning behind the three words, he used a mathematical structure called group and a system of kinship developed by Claude Lévy-Strauss as his narrative examples. The three aspects later have become a focus in discussions on structuralism.

Among works on structuralism based those three laws, I find the book by Cherryholmes most close to what I am looking for. There are two pieces of text in the book which are relevant. The first piece is

Structuralism is a method of study that has been used to describe and analyze phenomena as widely disparate as literature, politics, economics, myths, and cultures. A key assumption about structurally examined phenomena is that they are characterized by an underlying structure, not too surprising, that is defined, in part, by relationships among their constitutive elements. Structuralism broadly considered, then, includes assumptions about structures and methods of analysis.

(Cherryholmes, 1988, p. 16).

The text indicates an areas where structuralism can be applied to or has been applied to, and at the same time lays the first assumption of structuralism. This assumption is a weak combination of other two texts, one by Terence Hawkes and other by Jonathan Culler. The text by Hawkes is
This new concept, that the world is made up of relationships rather than things, constitutes the first principle of that way of thinking which can properly called 'structuralist'. At its simplest, it claims that the nature of every element in any given situation has no significance by itself, and in fact it determined by its relationship to all the other elements involved in that situation.

(Hawkes, 1977, p. 18).

and the text by Culler is “Thus structuralism is based, in the first instance, on the realization that if human actions or productions have a meaning there must be an underlying system of conventions which makes this meaning possible.” (Culler, 1973). In the same book, Cherryholmes combined the preceding two texts with the three aspects of Piaget, and offered the following definition of structuralism.

Structural analysis, · · · emphasizes wholeness and totality, not units and parts. The focus on wholeness comes from concentrating on systemic relationships among individual elements, not on their unique characteristics. Structural analysis also deals with transformations. If a structure is determined by relationships among its units, then those relationships, if the structure is to survive, must regenerate and reproduce the structure. Furthermore, structures are self-regulating, their relationships governing which activities are and are not permitted.

(Cherryholmes, 1988, p. 18).

Cherryholmes’ texts offer a meaning on structuralism which I am looking for. But now I have to decide on a way of presenting this meaning which should have clear character and distinct character. Among all choices, I feel an axiomatic way mostly suitable. I shall choose this way not for the rhetorical sake, but for the sake of being susceptible of interpretation. The axiomatic way of defining structuralism made here is the same as the axiomatic definition of points and lines made by Hilbert (1902). The following is a definition on structuralism which I shall use throughout the essay.

The structuralism consists of two parts — structure and method. Altogether four axioms are imposed on them. Two axioms are on structure while other two on method.

Axioms on structure:

• Structure is ontological and is susceptible of interpretation. Structure has two sides — general and particular, or what amounts to the same thing as global and local. The general side is an active side.

• Interrelationships between special cases of structure are in focus, not special case itself.

Axioms on method:

• Method is operational and it transforms structure into structure.
• Given special cases of structure can transform into new special cases of structure, and such possibility depends on the form which the given special cases are in.

In conformity with a general usage, the method is also called the structural analysis or the analysis on structure. By this I have completed a definition of structuralism. And now I shall interpret or realize the so-defined structuralism by some examples.

The first example is to interpret structure as water

\[ \text{structure} = \text{water}. \]

Special form of water can be a cup of water, water in a reservoir, or a piece of ice. A cup of water can transform into a piece of ice under a suitable condition. By pouring several cups of water into another cup, a new cup of water is formed.

The second example is to interpret structure as functions

\[ \text{structure} = \text{functions}. \]

General forms of function are \( f(x), g(x) \), while special forms of function are

\[ x^2 + 1, \sin x, \cos x, e^{3x}. \]

By the addition and the multiplication, functions transform into functions. The investigation on the way special functions transform into general functions lead to the well-established discipline called Harmonic Analysis. In this discipline one can prove that

\[ f(x) = a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots \]

holds for any general periodic function \( f(x) \).

The third example is to interpret structure as syllogisms in mathematics education

\[ \text{structure} = \text{syllogisms}. \]

A general syllogism here is a unitary syllogism defined by when one teaches one is taught. We have seen several special syllogisms and the way special syllogisms transform into others in the preceding chapter. In conformity with interpreting structure as syllogisms, the structural analysis here becomes the syllogistic analysis.

The structuralism formulated in this section contains thus non-trivial examples, and the examples in the last chapter are examples of the syllogistic analysis. With these words I end now this section.
4.2 Nicholas Bourbaki

In the summer 1935 some young French mathematicians formed a group to which the name Nicholas Bourbaki was given. The core figures in the group were André Weil, Jean Dieudonné, Henri Cartan, Claude Chevalley and Jean Delsarte. An ambition of Bourbaki was to find a boundary of human mathematical reason by building a mathematical edifice from scratch. Indeed Bourbaki wrote: “My efforts during the last fifteen years (⋯) have been directed wholly towards a unified exposition of all the basic branches of mathematics, resting on as solid foundations as I could hope to provide.” (Bourbaki, 1949). Thus, Bourbaki’s point of emphasis was at that time not on creating mathematics but on a method of recording mathematics. This position immediately put Bourbaki’s project on trial. Mathematics is a part of logic, if the emphasis is on the way of presenting mathematics rather than mathematics itself then what happens to the logic behind mathematics presented? Is it possible that the presentation is made for the rhetorical sake? For this replied Bourbaki with: “Whether mathematical thought is logical in its essence is a partly psychological and partly metaphysical question which I am quite incompetent to discuss. On the other hand, it has, I believe, become a truism, which few would venture to challenge, that logic is inseparable from a coherent exposition of the broad foundations on which mathematical science must rest.” (Bourbaki, 1949). Here emerges Bourbaki’s contention on philosophy of mathematics which should have far-reaching consequences in mathematics education. The contention is

- A presentation of mathematics is a part of mathematics.
- The presentation must take philosophical factors into consideration.
- The presentation must take psychological factors into consideration.

Bourbaki’s philosophy of mathematics does not come from the middle of nowhere, in fact it was strongly influenced by Euclid (Heath, 1908) and Hilbert (1902); which, in turn, are accumulation of works done by countless scholars during a period of thousands of years. I need to emphasize that it was Euclid who made the first attempt in that direction and it was then Hilbert who transformed that direction into a main stream of entire mathematics. What just said is the most fundamental part of Bourbaki’s philosophy of mathematics, and it was expounded in greater depth in two articles (Bourbaki, 1949) and (Bourbaki, 1971). In which argues Bourbaki:

Proofs, however, had to exist before the structure of a proof could be logically analyzed; and this analysis, carried out by Aristotle, and again and more deeply by the modern logicians, must have rested then, as it does now, on a large body of mathematical writings. In other words, logic, so far as we mathematicians are concerned, is no more and no less than the grammar
of the language which we use, a language which had to exist before the grammar could be constructed.

\[
\ldots\ldots
\]

Logical, or (what I believe to be the same) mathematical reasoning is therefore only possible through a process of abstraction, by the construction of a mathematical model. Every such step involves, in other words, an application of mathematics to something of a different nature.

\[
\ldots\ldots
\]

The primary task of the logician is thus the analysis of the body of existing mathematical texts, particularly of those which by common agreement are regarded as the most correct ones, or, as one formerly used to say, the most “rigorous.” In this, he will do well to be guided more by what the mathematician does than by what he thinks, or, as it would be more accurate to say, by what he thinks he thinks; for the mental images which occur to the working mathematician are of psychological rather than logical interest.

(Bourbaki, 1949, pp. 1–2).

Thus, the two articles become a blue-print for the project of building a mathematics edifice. The result of the project is a series of books by the name *Éléments de Mathématique*, the first 38 volumes were published by Hermann and the last 3 volumes by Masson. A guiding principle of writing that series is a strong belief that not only mathematical knowledge should be formalized but also it indeed can be formalized.

As every one agrees today, the distinctive character of a mathematical text is that it can be formalized, i.e., translated into a certain kind of sign-language. The first thing I have to do is therefore to lay down the vocabulary and grammar of the sign-language I wish to use, in its pure form at first, and later with all the modifications which usage has taught us to be required.

(Bourbaki, 1949, p. 3).

If a single word is picked up to describe Bourbaki then that word is *formalization*. The formalization means to record knowledge by certain language. A formalized knowledge in Bourbaki is thereby stored in certain sign-languages not in any natural language. Choice of that language depends not only on mathematics but also on philosophy and psychology. The chosen language should conform with philosophical consideration on the one side, and supplies with psychological safety on the other side.

Formalized knowledge is on a piece paper and is independent of material things. It is valid at any time, at any place, at small scale as well as at large scale. The knowledge about an ice-cube described on a piece of paper is safer than that observed by just looking at it in a hot summer day. Two other examples of the same kind were narrated by Dutch physicist Gerard ’t Hooft.
Let us begin our journey to the world of the tiny by beginning with what we can see with the naked eye, and with those laws of physics that we are all accustomed to. Take a large piece of paper and fold it into an airplane. You may decide to cut the paper in half to make two smaller airplanes. You could even cut the smaller pieces again to make even smaller planes. The properties of the paper, and the rules for folding it into an airplane, will hardly be different, except that the airplanes will become smaller and smaller. Gradually, however, as you continue to cut the paper into smaller and smaller pieces, you will find it harder to make planes, and eventually you will only have little shreds of what once was usable pieces of paper. The property ‘foldable into an airplane’ has been lost.

We have a similar situation when we start off with a bucket of water and pour the water into smaller buckets. The physical properties of water such as ‘flows from high to low’ remain the same, until finally we have less than one drop left. You cannot pour drops from high to low; you have to shake them off.

(Hooft, 1997, p. 1).

In formulation of laws in physics it is singularly urgent to formalize them in terms of mathematical knowledge. In addition those most safe laws must be formalized by mathematical equations, and it is simply not enough to formalize these laws by mere mathematical terminology. As American geometers Yau and Nadis say:

Mathematics is often called the language of science, or at least the language of the physical sciences, and that is certainly true: Our physical laws can only be stated precisely in terms of mathematical equations rather than through the written or spoken word. Yet regarding mathematics as merely a language doesn’t do justice to the subject at all, as the word leaves the erroneous impression that, save for some minor tweaks here and there, the whole business has been pretty well sorted out.

In fact, nothing could be further from the truth. Although scholars have built a strong foundation over the course of hundreds—and indeed thousands—of years, mathematics is still very much a thriving and dynamic enterprise. Rather than being a static body of knowledge (not to suggest that languages themselves are set in stone), mathematics is actually a dynamic, evolving science, with new insights and discoveries made every day rivaling those made in other branches of science, though mathematical discoveries don’t capture the headlines in the same way that the discovery of a new elementary particle, a new planet, or a new cure for cancer does.


On the methodological side Bourbaki is a structuralist in the sense given in the last section, for the method in him is the structural analysis. Principal objects in Éléments are mathematical structures and a principal aim is to order interrelationships among those mathematical structures. A hierarchy is introduced into the structures. At the very bottom lie three structures called mother structures. The first of these mother structures is the algebraic structure, the second is the order structure, and the third is the topological structure.
The most important algebraic structure is the group structure which was already introduced in the last chapter in the additive context. Now I shall express it in a slightly different way. The group structure is a binary operation satisfying four requirements. A binary operation is a binary syllogism, for the binary operation $a \cdot b = c$ can be looked upon as a synthesis $c$ of $a$ and $b$, namely,

\[
\begin{array}{c}
c = a \cdot b \\
a \\
\downarrow \\
b
\end{array}
\]

Likewise, the metric structure, the simplest topological structure, can be expressed syllogistically as

\[
\begin{array}{c}
c = \varrho(a, b) \\
a \\
\downarrow \\
b
\end{array}
\]

where $\varrho(a, b)$ is the distance between $a$ and $b$. And the order structure is expressed as

\[
\begin{array}{c}
c = O(a, b) = \pm 1 \\
a \\
\downarrow \\
b
\end{array}
\]

If the order $O(a, b) = 1$ then we say $a$ is greater than $b$, and if $O(a, b) = -1$ then we say $a$ is less than $b$. The group structure leads to groups, the topological structure leads to topological spaces, and the order structure leads to ordered spaces. These are the fundamental objects in *Éléments* where focus is on interrelationships among them. Thereby *Éléments* is the syllogistic analysis in mathematics.

By the 1950s not only Bourbaki and all its members were full-fledged and established their fame internationally but also *Éléments de Mathématique* had become one of the major university textbooks. The books had educated a generation of prominent mathematicians to which belongs Swedish mathematician Lars Hörmander who wrote the following lines:

> It was then that I started to read the Bourbaki books. For many years I continued to read the new volumes as they appeared, and they were an essential part of my mathematical education as for many mathematicians of my generation.

(Hörmander, 2000, p. 406).
The 1950s and 1960s were glorious days for Bourbaki and its *Éléments*. Bourbaki’s thought of unification in terms of structural analysis prevails everywhere in mathematics, and parts of mathematics that can not be unified with this frame were side-pushed. It was fashionable to work on big structures in the spirit of Bourbaki, and works on concrete problems were not given enough attention. Between 1950 and 1966 twelve young mathematicians were awarded Fields Prize among them four were French. Three of them, Laurent Schwartz, Jean-Pierre Serre and Alexander Grothendieck, were members of Bourbaki; although René Thom was not a Bourbaki, in fact he was almost an anti-Bourbaki, but his works were influenced strongly by Henri Cartan and Charles Ehresmann who were senior members of Bourbaki. One of requirements in getting Fields Prize is that the age of laureate is not over forty, this is probably a reason that none of founders of Bourbaki got this honor. However Henri Cartan and André Weil were awarded Wolf Prize around 1980 for their life-long achievements in mathematics.

The social life of Bourbaki during this period was extraordinarily active as well, and members of Bourbaki were invited to every major university cross the world to lecture on their doctrines. Bourbaki was a star. Like any other stars, not only their being and doing were admired but also they were copied. In Sweden for instance a group of mathematicians ganged up and called themselves H. Gask and wrote a book *Ordinära Differentialekvationer* which was rendered into English in 1973. Following the pattern of unification by structural analysis set forth by Bourbaki a large amount of textbooks were published during this period. Any unification can only be appreciated from a global perspective and appreciation of the achievements done by Bourbaki preconditions overall knowledge of its *Éléments* which has some forty volumes. Is this possible for an average university student? The answer is clearly negative. And so, any reading Bourbaki is bound for being local and thereby leads to a compartmentalization of knowledge which is trivially opposite to what Bourbaki had intended. Ironical, isn’t it? This situation was in fact not just restricted to reading Bourbaki and was quite general in at least American universities. The concern was spelled out by Singer and Thorpe in their book.

At the present time, the average undergraduate mathematics major finds mathematics heavily compartmentalized. After the calculus, he takes a course in analysis and a course in algebra. Depending upon his interests (or those of his department), he takes courses in special topics. If he is exposed to topology, it is usually straightforward point set topology; if he is exposed to geometry, it is usually classical differential geometry. The exciting revelations that there is some unity in mathematics, that fields overlap, that techniques of one field have applications in another, are denied the undergraduate. He must wait until he is well into graduate work to see interconnections, presumably because earlier he doesn’t know enough.

(Singer and Thorpe, 1967, p. v).
The formalization of mathematics by sign-languages was also advocated by Frege, Peano, Hilbert, Russell, Whitehead and many others; it is a part of mathematics — that part on paper. Exclusive formalization leads necessarily to totalization which is fatally destructive; for anything, of not seeing its own down-side, is unavoidably going to evolve, by its own power, into its own stationary. The formalizations carried out by Frege, Russell and Bourbaki excluded intuition and application in which dialectic of these formalizations exist. This is particularly clear in the case of Bourbaki. Indeed by the 1970s Bourbaki’s ignorance of probability, statistics, applied mathematics as well as numerical mathematics and its powerlessness on many concrete problems became increasingly evident. This, together with the fact that all its founders had reached the age of retirement, prepared a coming stationary period of Bourbaki. Today we are still in this period.

In the 1960s not only Bourbaki was seen scientifically and socially, but also the shadow of him was seen in higher schools through curriculum. A catch word in the 1960s is revolution. There were all kinds of revolution, and revolutions can take place anywhere on the earth. The most famous revolutions in the 1960s are the rock and roll revolution, the sex revolution, the cultural revolution, and so on. A revolution usually starts in reason but soon the reason is taken over by enthusiasm, and a revolution often carries political marks. In a revolution there is always a distinction between the old and the new with the old being evil and the new virtue. It is a main purpose of a revolution to push aside or even to demolish by revolutionary power the old in order to experience the new. This very fact that the old can only be removed by a revolutionary power which often is violent is in itself a dialectic of great vitality of the old. This accounts for the necessary recurrence of the old in the period of post-revolution. The New Math Movement was a revolution indeed a revolution in mathematics education. The political impetus came probably from the launch of Sputnik I in October of 1957 on the one side and the consequential formation of School Mathematics Study Group in the spring of 1958 in USA on the other side. It is easy to point out new things in New Math, for in it everything is new. It is not just to dress up the old things as the new things by some new clothes, everything is genuinely new inside out. Those new things are made of sets by the structural analysis of Bourbaki, and consequently Bourbaki becomes a godfather of the New Math and Theory of Sets its very foundation. This is reasonable, isn’t it? Isn’t it Bourbaki who tried to start with sets and then built a whole mathematics edifice by some structures on sets? Isn’t it Bourbaki who wrote *Éléments de Mathématique* which educated a generation of distinguished mathematicians? Thought so first and did so then. After just a few years of preparation, an introduction of the New Math into schools became a fact.

I believe a successful mathematics education must take three factors into consideration. The first one is the classroom instruction, the second is the social network and the third is the self-instruction. The classroom in-
struction is an interaction among teachers, pupils, curriculum material and textbooks in a classroom, and the social network usually consists of classmates and close relatives. A good introduction of the New Math demands a good teacher education prior to the introduction, and a sudden introduction does not take this into consideration. It is therefore not difficult to imagine what feels like for an average pupil facing sets and not getting enough good instruction from teachers as well as from relatives. It is just a matter of time when fatal disadvantage of the New Math was revealed. The time needed was not long, just about ten years.

The aim of this discussion on the New Math Movement in a section on Bourbaki is not to put a blame on Bourbaki for unsuccessful trial of the New Math. In fact it was not Bourbaki who wanted a such trial in the first place. His contention that a major part of mathematics is structural analysis is still valid to these days. I am certain that Bourbaki shall return to mathematics education, in perhaps another form.

4.3 Syllogism

A syllogism is in the broadest sense a relationship among a thesis, an antithesis and a synthesis. Traditionally, it is written as

\[ \text{thesis} \rightarrow \text{antithesis} \rightarrow \text{synthesis}. \]

This way of writing makes the relation between thesis and antithesis asymmetric. But sometimes a symmetry between them does appear, and then in that case I write a symmetric syllogism in the following fashion:

\[ \begin{array}{c}
\text{synthesis} \\
\text{thesis} \\
\text{antithesis}
\end{array} \]

In conformity with this way of writing, I shall at times write an asymmetric syllogism also in terms of a triangle but with oriented sides.

A purpose with syllogisms is to describe a general pattern of nonlinear development

\[ \begin{array}{c}
\text{thesis} \\
\text{synthesis} \\
\text{antithesis}
\end{array} \]

in comparison to a linear development
The nonlinear development described above is much more general than the linear one, and such development is a dominate one from the historical perspective. A well-known Chinese proverb: “a river flows along eastern bank for thirty years and then along western bank for thirty years” describes just such development.

As it was said, a syllogism is a relation among three things, but such a general syllogism is not the concern of the present essay. Thereby I shall first of all make some determination on the word *syllogism*.

The easiest way of defining syllogism is via the concept of function in mathematics, in particular the concept of function in two variables. Mathematically, a function in two variables is a relation among three things \( x, y, z \). In such a relation \( x \) and \( y \) are free variables but \( z \) is a variable which depends on \( x \) and \( y \). This means that when values of \( x \) and \( y \) are given, then a value of \( z \) is consequently determined. For such a function we write

\[
z = f(x, y).
\]

For instance \( z = f(x, y) = x^2 + 2y \) in which case if \( x = 2 \) and \( y = 1 \) then \( z = 6 \), that is \( z = 6 \) is a synthesis of \( x = 2 \) and \( y = 1 \) via the function \( z = x^2 + y \). Generally, we can also interpret a function \( z = f(x, y) \) as a synthesis \( z \) of \( x \) and \( y \) and write

\[
\begin{array}{c}
z = f(x, y) \\
x \\
y
\end{array}
\]

In my definition of syllogism I shall lean on the concept of function in two variables, thereby I say that a syllogism is a function (synthesis) of a thesis and an antithesis. Such a function, unlike a function in mathematics, is generally multi-valued. Thus, in a syllogism, a thesis and an antithesis are prescribed, then by certain way a synthesis is produced, not necessarily in unique way. Sometimes I use the word *side* for thesis or antithesis or synthesis. With this terminology a syllogism is a function (the third side) of other two sides.

Syllogisms consist in act of thinking and thereby appear in any process of cognizing. Insofar as cognition is involved *reason* must be taken into consideration, for reason, as a human faculty, is a source that makes cognition possible. Inasmuch as the essay is not philosophical I shall not define exactly the meaning of *reason* instead of use a common understanding on that word via common sense. The inquiry into reason and structure of reason have been being the most challenging enterprise and the most wanted dream of the entire philosophy, sometimes even psychology. It was discovered at very early stage of the inquiry that a certain cognitive force exists in reason, the force that attempts to group things together in certain definite fashion. It is
this force that is a substratum of nearly all structures in reason discovered so far. Instinct and logic are such structures in reason, although they are of different characters. A function of the former is to survive or what amounts to the same thing as to make life possible, and the latter is to make life better.

In instinct; hunger and eating are grouped together, running and chasing (in the animal world) are grouped together, raining and umbrella are grouped together, and so on. We write them symbolically

\[
\text{hunger} \rightarrow \text{eating} \\
\text{running} \rightarrow \text{chasing} \\
\text{raining} \rightarrow \text{umbrella}
\]

and grouping here describes a reaction on a given action. A formula for this sort of grouping is

\[
\text{stimulus} \rightarrow \text{response}.
\]

In logic grouping is another kind. Typical examples are

\[
\text{if } \begin{cases} A \text{ is } B \\ B \text{ is } C \end{cases} \text{ then } A \text{ is } C
\]

which is the Aristotelean syllogism and

\[
\text{counting} \rightarrow \text{de-counting} \rightarrow \text{re-counting} = \text{measuring}.
\]

Grouping of two objects in instinct and grouping of three objects in logic have been considered explicitly or implicitly as a lower structure and a higher structure in reason. This theoretical position is particularly true in a part of developmental psychology created by Vygotsky. I shall return to this issue in the next chapter. As the title of the essay indicates, a principal concern here is on grouping of three objects, namely syllogism, I shall thereby put the attention almost exclusively on it from now on.

As it was said above, a syllogism is such a relation among three sides that the third side is conceived when the first two sides are given. Syllogisms thereby can be classified according to the way of giving the first two sides. If the first two sides are given explicitly, by which I mean the two sides are related to each other in simple way in reason, then the syllogism is said to be binary. If on the other hand, the first two sides are given implicitly through some complex relation in reason, then the syllogism is said to be unitary. Since a clear boundary between simplicity and complexity is missing, thereby that between a binary and a unitary is not completely clear. In a practical use of those two words a possible overlapping could not completely avoided.

Except for the example given above, the following examples are typical unitary syllogisms:

\[
\text{centralization} \rightarrow \text{de-centralization} \rightarrow \text{re-centralization},
\]
equilibration $\rightarrow$ de-equilibration $\rightarrow$ re-equilibration.
The general formula for the unitary syllogism is

$$A \rightarrow \text{de-}A \rightarrow \text{re-}A = A\ 	ext{sublated.}$$

Unitary syllogisms are rooted deeply in reason, and is produced by the *cunning of reason* which shall be discussed in a later chapter.

The most important examples of binary syllogism in mathematics are no doubt the group structure, the metric structure as well as the order structure mentioned in the section on Bourbaki. A syllogisms whose first sides are being and doing, necessity and contingency, intuition and formalization, general and particular are all examples of binary syllogism. In the last chapter other examples were observed as well, for instance, the syllogism whose first two sides are the side of five is five books and the side of five is five apples, and a syllogism (an equation) whose first two sides are constants and variables.

The essence of a syllogism is to be able to look upon an thesis in combination of its antithesis (a dialectic of thesis), in plain words, to be able to look at a thing from both its positive side as well as from its negative side. Consequently, the knowledge obtained is in one way or other a piece of comparative knowledge. The shift from ontological knowledge to comparative knowledge is reminiscent of the shift from Newtonian Mechanics to Theory of Relativity.

It was stated that Aristotle discovered logic some two thousands years ago and it was Frege who rediscovered it in his *Begriffsschrift* published in 1879 (for English translation see Heijenoort, 1967). During a period of nearly two thousands years from Aristotle to Frege, logic, as a deductive science, was completely dominated by the Aristotelean syllogism. This position can be seen clearly from two different translations of the same piece of text of Aristotle. The first translation is recorded in (Aristotle, 1984). It is this work I refer to when I talk about Aristotle, not others.

All teaching and all intellectual learning come about from already existing knowledge. This is evident if we consider it in every case; for the mathematical sciences are acquired in this fashion, and so is each of the other arts. And similarly too with arguments—both deductive and inductive arguments proceed in this way; for both produce their teaching through what we are already aware of, the former getting their premises as from men who grasp them, the latter proving the universal through the particular’s being clear. (And rhetorical arguments too persuade in the same way; for they do so either through examples, which is induction, or through enthymemes, which is deduction.)


The second translation was recorded in (Aristotle, 1928). Pages in the entire volume are not numbered, but the following translation can be read off on the first page of the article *Analytica Posteriors*. 
ALL instruction given or received by way of argument proceeds from pre-existent knowledge. This becomes evident upon a survey of all the species of such instruction. The mathematical sciences and all other speculative disciplines are acquired in this way, and so are the two forms of dialectical reasoning, syllogistic and inductive; for each of these latter make use of old knowledge to impart new, the syllogism assuming an audience that accepts its premises, induction exhibiting the universal as implicit in the clearly known particular. Again, the persuasion exerted by rhetorical arguments is in principle the same, since they use either example, a kind of induction, or enthymeme, a form of syllogism.

(Aristotle, 1928).

In comparison of these two translations we take notice that deduction in Barnes’ translation becomes syllogism in Mure’s. However both translators use the same word enthymeme which is an abridged syllogism. Taking consideration of a dominating role played by syllogism in Aristotle’s organon, it is clear that for Barnes deduction is syllogism in Aristotle. However in a contemporary usage of these two words, deduction has a bigger category than syllogism and I shall always make a distinction between them. In addition, in the Chinese translation of Complete Works of Aristotle, syllogism is used. Put aside the differences of the translations, at least one thing is clear, for Aristotle, scientific knowledge has to be constructed by either induction or deduction and a piece of deduction is a piece of syllogism. This thesis of Aristotle later became a target of attack by Francis Bacon and René Descartes to which I shall come back with a discussion in a later chapter. Even under the attack and even under the intervention by symbolic logic inaugurated by Frege, the deductive sciences with syllogism as their principal tool continued to develop all the way into the twentieth century. In the hands of logicians from the Polish school of logic, they became one of the most important part of modern mathematical logic (see Łukasiewicz, 1957; Tarski, 1994).

The way of constructing knowledge by induction differs a great deal from that by deduction. In fact the difference is so great that Aristotle had to make distinction between them (see the quotation above). There seems to exist, side by side, two kind of sciences — deductive sciences and inductive sciences. If that is the case, then reason which is responsible for sciences consists of two irreducible constituents. The one is the ability of apperception and the other is the ability of deducting. At operational level, a trait of induction and that of deduction differ also much, for instance, it is relatively easy to formalize a piece of deduction in comparison to formalizing a piece of induction. In fact I have not yet seen any satisfactory formalization of induction. Inspection on the history of logic reveals that the development of logic is almost exclusively on deduction. Mathematics is definitely a deductive science and the consequential education has its emphasis on deduction. An educational concentration on deduction is equivalent to excluding for instance applied mathematics education, for knowledge in such science
can only be constructed by alternating induction and deduction. Einstein’s derivation of Brownian motion and Dirac’s derivation of the fundamental equations in quantum mechanics are salient samples of such knowledge.

Reasoning is a process in which a certain thing, called side, is asserted based on other things also called sides. If the number of these other sides is two then the reasoning is a deductive (syllogistic) reasoning; if that number is genuinely greater than two then the reasoning is an inductive reasoning. When the number of sides is very large, it is practically impossible for reason to take each side into consideration, thereby if there is any thing to be asserted then that thing is of average in essence. This means that inductive reasoning is bound up with statistical reasoning, thereby apperception is a substratum of statistical reasoning.

4.4 Sides

Now that we know our didactical thought is to perform the syllogistic analysis on pieces of mathematics which consists in interrelationships among syllogisms. Each syllogism consists of three sides, thereby it becomes an interesting question of what is the general form of sides in a syllogism appearing in mathematics education. In searching for such a general form we must of course have some criteria. The first criterion is that the form in searching should take intuition into account and can be formalized. The second criterion is that the form should contain the form of sides in an Aristotelean syllogism as a special case.

We begin with the easy one. As it was showed above that all three sides in an Aristotelean syllogism has the form of $A$ is $B$. That of $A$ is $B$ is not only a statement but indeed a logical assertion. What this really means is that $A$ is $B$ in logic. Let us express this by the following diagram:

$$A \xrightarrow{\text{logic}} B$$

which clearly conveys the information of $A$ is $B$ through logic. I shall compare this formulation with the method of coordinate in geometry.

Objects in geometry are points, lines, surfaces, and so on. Objects in algebra are numbers, equations, and so on. It is through systems of reference that a point relates itself to a aggregation of numbers. In three dimensional case, a point $P$ relates via a system of reference to three numbers $x, y, z$. Symbolically we write

$$P \leftrightarrow (x, y, z).$$
There are two versions to express this fact. The weak version is to say of the coordinate of a point $P$ is $(x, y, z)$, or the strong version is to say of a point $P$ is a coordinate $(x, y, z)$ in that system of reference. A similar diagram is

\[ P \rightarrow (x, y, z) \]

system of reference

And in this case the place of $P$ and that of $(x, y, z)$ can be interchanged.

Now we turn to intuition. Since I use this word very limitedly, thereby I shall not define the meaning of the word, instead of I pick up the meaning by common sense. When a person observes the world by intuition, at the most fundamental level, the person has to assert a relation between two objects $A$ and $B$. Especially the person has to assert whether $A$ is $B$. Assume the person asserts $A$ is $B$ in this way. Then the value of the assertion of $A$ is $B$ lies completely in intuition, for without intuition any relation between $A$ and $B$ can not be asserted. Let us express this by a similar diagram:

\[ A \rightarrow B \]

intuition

Hence, in any cognitive science, a relation between $A$ and $B$ is needed to be established. If $A = B$ then it holds the tautology of $A$ is $A$, in which there is no cognitive act. If $A \neq B$, then a relation between $A$ and $B$ is meaningless unless a context $C$ is being prescribed. Insofar as a relation between $A$ and $B$ is concerned, it is a relation in a context $C$, in which case $A$ is a dialectic of $B$ in that context $C$, and vice versa.

This motivates the following consideration. Given two objects $A$ and $B$. We shall call an ordered pair $(A, B)$ a cognitive state or simply a state and shall write a cognitive state as

\[ A \rightarrow B. \]

It follows from the discussion made above that the assertion of $A$ is $B$ in logic and the assertion of $A$ is $B$ in intuition are examples of cognitive state. The assertion of the coordinate of the point $P$ is say $(1, -3, 0)$ is another example. It is very convenient sometimes to render the word *is* into the sign $\rightarrow$, so that an assertion of $A$ is $B$ is rendered into $A \rightarrow B$. 


The simplest cognitive state is no doubt the tautology of \( A \) is \( A \) and the question of \textit{what is} \( A \), or what amounts to the same thing as

\[
A \rightarrow A \\
A \rightarrow \text{what}
\]

I sometimes also call the question of what is as the what-law.

A cognitive state \( A \rightarrow B \) is a result of action of thinking on the question of what is \( A \), thereby a partial answer to that question. The \( B \) consequently brings out certain properties of the \( A \), in other words, \( B \) is a determination on \( A \).

Let us take an example of \( A = 5 \). We have the following cognitive states:

\[
\begin{align*}
5 & \rightarrow \text{what}, \\
5 & \rightarrow \text{five cows}, \\
5 & \rightarrow \text{five books}, \\
5 & \rightarrow x \text{ such that } x + 4 = 9, \\
5 & \rightarrow x \text{ such that } x^2 - 6x + 5 = 0, \\
5 & \rightarrow \text{the maximum of the function } 2x - 1 \text{ where } 0 < x \leq 3.
\end{align*}
\]

The cognitive state \( A \rightarrow A \) is a state produced without much of thinking. No one can deny the innocence and genuineness of this state. However if we restrict ourselves only to this kind of state then we stay within one perspective and exclude interaction between different perspectives. For instance if we would restrict ourselves only to the state of daughter is daughter and the state of mother is mother then we would not be able to find any relationship between a daughter and a mother. Such a relationship exists in interaction of those two states, and the interaction has different forms. A daughter is a person that a mother gives birth to, and a mother is a person who gives birth to a daughter. However, in epistemology, the state \( A \rightarrow A \) makes the beginning; and in this state \( A \) manifests in itself, or in another word, at this stage, \( A \) is a thing-in-itself. In the next stage \( A \) turns outwards, the state \( A \rightarrow A \) develops into a state \( A \rightarrow B \), and at this stage \( A \) is a thing-for-others. When cognition is complete then the \( A \) is a thing-for-itself. When \( A \) is a thing-for-itself the state \( A \rightarrow A \) is sublated into itself.

Now we return to the query of general form of sides in a syllogism appearing in mathematics education, and come to a conclusion. The general form of sides is a general form of cognitive states, that is \( A \rightarrow B \). Thereby the general form of syllogisms appearing in mathematics education is

\[
\begin{align*}
C & \rightarrow D \\
A_1 & \rightarrow B_1 \\
A_2 & \rightarrow B_2
\end{align*}
\]
where \( A_1 \rightarrow B_1 \) and \( A_2 \rightarrow B_2 \) are the first two sides and \( C \rightarrow D \) is the third side. We represent this fact symbolically as

\[
(A_1 \rightarrow B_1) + (A_2 \rightarrow B_2) = (C \rightarrow D).
\]

The equation supplies the third \( C \rightarrow D \) with a special role, for it is produced by other two sides. This makes the preceding syllogism asymmetric. The symmetry here means that not only the third side is produced by other sides, but also the first side or the second side is produced by other sides. In symbols, the equations hold

\[
(A_1 \rightarrow B_1) = (C \rightarrow D) - (A_2 \rightarrow B_2)
\]

and

\[
(A_2 \rightarrow B_2) = (C \rightarrow D) - (A_1 \rightarrow B_1).
\]

If the preceding syllogism has such symmetry then we say that the syllogism is stable.

Let us take a look at two examples of stable syllogism and see some implication of the stability and these examples should be compared with (Piaget, 1985, pp. 7-21).

The first example is from the last chapter which is \( I + II = III \)

The three sides of this syllogism are

- five is five books (I)
- five books is five apples (II)
- five is five apples (III)

The stability of the syllogism is

\[
I + II = III, \quad III + (-II) = I, \quad (-I) + III = II.
\]

Hence the sides I and II must be reversed in the sense of

- five books is five (\(-I\))
- five apples is five books (\(-II\))
The equation \( III + (-II) = I \) means a commutative diagram

\[
\begin{array}{c}
\text{five} \\
\downarrow \\
\text{five books} \\
\uparrow \\
\text{five apples}
\end{array}
\]

and the equation \( III + (-I) = II \) means a commutative diagram

\[
\begin{array}{c}
\text{five} \\
\downarrow \\
\text{five books} \\
\uparrow \\
\text{five apples}
\end{array}
\]

In an assertion of \( A \) is \( B \), the latter is larger category than the former. If, in addition, \( B \) is also \( A \), then \( A \) and \( B \) are equivalent in certain sense. Thereby, the \( II \) together with the \(-II\) establish the equivalence between the set of five books and the set of five apples in the sense of cardinal. And the \( I \) together with the \(-I\) establish the equivalence between the number five and the set of five books in a unification of mind and body.

The second example is the syllogism \( 5 + 7 = 12 \) showed in this chapter:

\[
\begin{array}{c}
5 + 7 = 12 \\
\downarrow \\
5 \\
\uparrow \\
7
\end{array}
\]

The stability of the syllogism means that \( 5 + 7 = 12 \), \( 12 - 5 = 7 \) and \( 12 - 7 = 5 \).
Chapter 5

The Cunning of Reason

Two theoretical underpinnings of this essay is the syllogistic analysis and the cunning of reason. The former is a point of emphasis of the preceding chapter, and the latter is that of the present one.

The syllogistic analysis concerns itself interrelationships among syllogisms, whereas the cunning of reason consists in a unitary syllogism itself. Originally, the cunning of reason is a philosophical category, for it originated in the Hegelian philosophical system. In fact, the cunning of reason has become an important constituent of that system. Not only so, the cunning of reason plays a decisive role in the Marxist enunciation of human development in a labour process, which, in turn, leads to Vygotskian thought on mastery of behavior from outside (the so-called mediated activity).

Mathematics education benefits from not only philosophy but psychology as well, for “Mathematics education is solidly grounded in psychology and philosophy among other fields” (Schoenfeld, 2008, p. 467). In addition mathematics education is intimately related to philosophy of mathematics. Thereby, before proceeding further, let me quote some descriptions on mathematics and on philosophy of mathematics. The first one is a general description given by Encyclopedia Britannica (2012).

Mathematics, the science of structure, order, and relation that has evolved from elemental practices of counting, measuring, and describing the shapes of objects. It deals with logical reasoning and quantitative calculation, and its development has involved an increasing degree of idealization and abstraction of its subject matter.

Philosophy of Mathematics, branch of philosophy that is concerned with two major questions: one concerning the meanings of ordinary mathematical sentences and the other concerning the issue of whether abstract objects exist.

(Encyclopedia Britannica, 2012).

Thus mathematics starts with counting and geometric shapes, and then organizes these empirical experiences into structures. Once structures are
formed, interrelationships enter on stage. As to the method that is used in an inquiry into interrelationships, wrote Courant and Robbins:

Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality. Though different traditions may emphasize different aspects, it is only the interplay of these antithetic forces and the struggle for their synthesis that constitute the life, usefulness, and supreme value of mathematical science.

(Courant and Robbins, 1996).

As to the two major issues within philosophy of mathematics, let us ask two simplified questions:

- what is the number 2?
- how is the number 2 constructed if it exists?

It seems that any philosophical inquiry into these two questions has not been completely fruitful. Answers are either too metaphysical to be of any mathematical use or else too controversial to be accepted at the large. However, answers given by Kant do contain certain essence, so that an important text shall be quoted here (style of boldface is from the original text):

Philosophical cognition is rational cognition from concepts, mathematical cognition that from the construction of concepts. But to construct a concept means to exhibit a priori the intuition corresponding to it. For the construction of a concept, therefore, a non-empirical intuition is required, which consequently, as intuition, is an individual object, but that must nevertheless, as the construction of a concept (of a general representation), express in the representation universal validity for all possible intuitions that belong under the same concept.


Even so, the two simplified questions are still too big to be considered here. Inasmuch as the essay is on mathematics education, I shall accept the existence of the number 2 and that of two books. Furthermore, I accept that the number 2 is an extension in mind and the set of two books is an extension in body. Under these circumstances I ask only one question of what is the relation between the number two and the set of two books. Extensions in mind constitute a world which is different from the material world (extensions in body) we physically live in. For the sake of argument I call this world the abstract world. The existence of the abstract world and the material world is almost taken as granted in all philosophical discussions, even though some formulation might be different. That everything has two perspectives — idealistic and materialistic — could be one of such formulations. Thereby philosophy progresses not by postulating such existence but by postulating interrelations between those two worlds. And such
interrelations are by no means static, indeed, dynamical development of a relation between the abstract world and the material world has been being at the center of philosophy, and it started right at the outset of the recorded philosophy and continues into our days.

5.1 A Philosophical Discussion

Philosophy, as once Bertrand Russell (1961, p. 13) described, is No Man’s Land between theology and science. Science is about definite knowledge appealing to human reason rather than to authority whereas theology is about dogma unascertainable to reason, and philosophy has traits of both science and theology. Since mathematics, as a generally accepted fact, is science, and I feel a need of writing down some words in defence of dragging philosophy into an essay on mathematics education.

First of all, mathematics is about mathematical knowledge and insofar as knowledge is concerned, we must inquire into form of knowledge, nature of knowledge, formation of knowledge, and so on. Secondly, we can benefit a great deal from such a philosophical discussion in obtaining methods which conduct our search for mathematical knowledge. Thirdly, mathematics on the other hand is often the first testing ground for philosophical hypothesis. At last, probably an obvious reason that I have to conduct a philosophical discussion is because of the cunning of reason which is one of two theoretical underpinnings of the present essay.

However I am aware of this is not a place to discuss philosophy in general and I therefore conduct my discussion on that part of philosophy which has direct bearing on my view towards mathematics education.

5.1.1 The World in Motion and the World in Form

If I pick up at random a piece of text on philosophy which concerns me and read it, then I probably read off the following keywords: knowledge, logic, concept, idea, soul, mind, body, matter, substance, reason, empirical, think, perceive, sense, a priori, a posteriori and so on. Of all these words knowledge plays a singular and important role in all philosophical discussions. These words might indicate some ideas on that part of philosophy I am in. Of course different philosophers might attach different meanings to these words, but principal meanings are almost unchanged. At very bottom they all in one way or other have something do with the way we see the world wherein we live. In short any philosophical discussion must begin with a world outlook, that is a view on the material world, the abstract world and their interrelationships.

At the present century where science is much advanced in comparison to the centuries wherein Plato, Hume and Kant lived; the existence of the material world is unquestioned. In addition there is overwhelming evidence
showing three fundamental forms of existence, and these are space, time and matter which in fact are united into motion. Motion is not eternal, it was born some billions years ago and shall die some billions years later. No one knows what had existed in the universe, if there is any, before the present one. Does a triangle exist there? During a period of thousands of years science and technology have advanced by leaps and bounds. New facts which are so familiar to us yet so unknown to philosophers lived before the 20th century make some famous philosophical debates uneven. We, equipped with the newest achievements of science and technology, can see plainly invalid arguments and incorrect premises made centuries ago by even some greatest philosophers. This does not make our discussion here valueless, for an aim of present discussion is not about the invalidness and the incorrectness. The aim here is merely to see the way that obstacles caused by the invalidness and the incorrectness are being overcome. For the sake of discussion I do not think it possible to continue some old philosophical discussions without making necessary modifications. These modifications must be based on modern scientific evidence. It seems at the first sight the modifications are few in rationalism against massive in empiricism, but scientific and technological revolutions achieved in the last century have made it urgent that fundamental modifications must be introduced into an overall philosophical discussion if such a discussion shall bear modern marks. These revolutions include Theory of Relativity, Quantum Mechanics, Information Technology, Genetics and DNA, to mention some of the most salient ones.

Nonetheless the most fundamental question remains almost the same. I shall formulate the question in the following fashion. Let $A$ be an object in our inquiry, say $A$ is a triangle. I shall attempt to inquire into three categories. The first one is the possibility of realizing $A$ in motion and the second one is the possibility of conceptualizing $A$ in reason. In short the first one is to put $A$ in motion while the second one is to put $A$ in head, and the $A$ in motion is the realistic $A$ while the $A$ in head is the idealistic $A$. The third category is interrelationship between an idealistic $A$ and a realistic $A$.

The recorded discussion on the abstract world, the material world and their interrelationship were set out by two of the greatest philosophers in the antiquity right at the beginning of western philosophy. Both Plato and Aristotle did not deny the existence of the two worlds, although they had different views on those worlds at the fundamental level. For Plato the material world is in constant motion and knowledge about this world is gained by perceiving, sensing and seeing with help of light. As a result a bulk of knowledge about the material world depends on location in space, on interval in time and on mass, and this kind of knowledge is susceptible of change and can not be trusted completely. In searching for knowledge which is independent of space, time and mass, Plato found another world, this world, as a real entity, is independent of motion and thereby can be trusted by human reason, and he expressed as “⋯ knowing what always
is, not what comes into being and passes away. · · · geometry is knowledge of what always is.” (Plato, 1997, p. 1143). In some sense the world Plato found is more real than the real world. This world is the abstract world, abstract in a sense that everything in this world is independent of motion therefore eternal. Substances of this abstract world are Forms which can not be seen by our natural eyes under the natural light, and they can only be seen by thinking with the help of light of mind. In Plato the light of mind is intelligence. As to the interrelationship between the abstract world and the material world, Plato used a metaphor in Book VII of Republic for the purpose. In fact Plato spent a large part of Book VII on this metaphor and its implications. I shall quote the beginning.

Imagine human beings living in an underground, cavelike dwelling, with an entrance a long way up, which is both open to the light and as wide as the cave itself. They’ve been there since childhood, fixed in the same place, with their necks and legs fettered, able to see only in front of them, because their bonds prevent them from turning their heads around. Light is provided by a fire burning far above and behind them. Also behind them, but on higher ground, there is a path stretching between them and the fire. Imagine that along this path a low wall has been built, like the screen in front of puppeteers above which they show their puppets.


One can of course argue that Plato accepted the existence of one world and at the same time denied that of the other. In that case he could simply have concentrated on the existed world, but it seemed that was not the case. Thing in itself can not supply it with the evidence on its own existence, but it can prove its existence indirectly by finding a dialectic of it. In Plato a dialectic is a shadow (the screen in Plato’s text in the quotation above). Thus in some sense Plato accepted the existence of two worlds, but the one consists of entities and the other consists of shadows. Thus, according to Plato, things in the material world are nothing but projected shadows of Forms in the abstract world. The ways of projection are restricted, so that things are only partial aspects of Forms and thereby can not be perfect. Insofar as shadows are mediation in knowing Form, knowledge about Form can not be completed and is always restricted by way of projection. This contention is expressed explicitly in a passage at the very end of Book VI of Republic.

Then you also know that, although they use visible figures and make claims about them, their thought isn’t directed to them but to those other things that they are like. They make their claims for the sake of the square itself and the diagonal itself, not the diagonal they draw, and similarly with the others. These figures that they make and draw, of which shadows and reflections in water are images, they now in turn use as images, in seeking to see those others themselves that one cannot see except by means of thought.

Already in the antiquity Plato’s world outlook was refuted by Aristotle. Before the refutation Aristotle wrote the following comment on Plato’s philosophy, and the text is from Book I of *Metaphysics*.

After the systems we have named came the philosophy of Plato, which in most respects followed these thinkers, but had peculiarities that distinguished it from the philosophy of the Italians. For, having in his youth first become familiar with Cratylus and with the Heraclitean doctrines (that all sensible things are ever in a state of flux and there is no knowledge about them), these views he held even in later years. Socrates, however, was busying himself about ethical matters and neglecting the world of nature as a whole but seeking the universal in these ethical matters, and fixed thought for the first time on definitions; Plato accepted his teaching, but held that the problem applied not to any sensible thing but to entities of another kind—for this reason, that the common definition could not be a definition of any sensible thing, as they were always changing. Things of this other sort, then, he called Ideas, and sensible things, he said, were apart from these, and were all called after these; for the multitude of things which have the same name as the Form exist by participation in it. Only the name ‘participation’ was new; for the Pythagoreans say that things exist by imitation of numbers, and Plato says they exist by participation, changing the name. But what the participation or the imitation of the Forms could be they left an open question.


A central point of the comment is about that projective relation between a material thing and its Form. In Plato true knowledge is about Forms and independent of the material world, and all we experience is nothing but shadows of Forms. No such knowledge can thereby explain phenomena such as cold, hot, big, small. On the contrary, Aristotle starts an inquiry right from the material world and reaches the abstract world by abstraction, as he put it in Book XI of *Metaphysics*.

as the mathematician investigates abstractions (for in his investigation he eliminates all the sensible qualities, e.g. weight and lightness, hardness and its contrary, and also heat and cold and the other sensible contrarieties, and leaves only the quantitative and continuous, sometimes in one, sometimes in two, sometimes in three dimensions, and the attributes of things *qua* quantitative and continuous, and does not consider them in any other respect, and examines the relative positions of some and the consequences of these, and the commensurability and incommensurability of others, and the ratios of others; but yet we say there is one and the same science of all these things—geometry),


The passage shows Aristotle’s insightful attitude towards mathematics and in fact contains a primitive thought of axiomatics. Contrary to that things in the material world are projections of Forms, Aristotle argued that Forms can only be attained by depriving of motion, in other words, Forms are end results of abstraction. In addition abstraction does not concern things
themselves but relative relations among things, these relative relations include quantitative relations as well as logical relations. Consequently things are insignificant but relations among things are in focus. This is the thought of axiomatics, and David Hilbert picked up this thought some two thousands years later and made it as a starting point of modern mathematics. Unfortunately Euclid failed to introduce this thought into his *Elements*, so that point and line mean nothing but point and line as such; while in the hands of Hilbert, point, line and plane at the outset mean point, line and plane but at the end their extension can include table, chair and mug.

The abstract world and the material world are not separated from each other, on the contrary, they are united in one world consisting of ideas and things. Let us call this world the universe. Being a unification of two worlds, each thing in the universe has a dialectic, so that a thing can be observed at least from two perspectives. As to how man acquires knowledge about the universe, Plato argued that man is a unification of body and soul, or what amounts to the same thing as body and mind. It is through the body that man perceives the material world, and it is through the soul that man senses the abstract world. Man, facing a new thing, always has a given perspective just like those human beings living in an underground, cavelike dwelling. Knowledge about that thing can be acquired by man only through liberating himself from the old perspective, by creating new perspectives and by comparing different perspectives. Creation of perspective and change of perspective are the focus of education. In fact it was already emphasized by Plato in Book VII of *Republic*.

Therefore, calculation, geometry, and all the preliminary education required for dialectic must be offered to the future rulers in childhood, and not in the shape of compulsory learning either. … Because no free person should learn anything like a slave. Forced bodily labor does not harm to the body, but nothing taught by force stays in the soul.


Here we see a typical example of progressive struggle between two nearly opposites. A progressive struggle or debate is constructive rather than destructive. It is an active force that drives a query forward instead of backward. A process of struggle is also a process of sublation in knowledge.

I now end this section by commenting Aristotelean logic and giving some examples on that logic.

Aristotelean logic, or Aristotelean formal logic, consists of a major part and a minor part, the former is deduction while the latter is induction. In fact the major part is so substantial that the entire Aristotelean formal logic is almost exclusively identified with deduction. A standard form in Aristotelean formal logic is the (Aristotelean) syllogism. In fact Aristotle developed his syllogistic logic so thoroughly that it took nearly two thousands years that new progress could be made. A typical Aristotelean syllogism consists
of a major premise and a minor premise and a conclusion, and it can be written symbolically as
\[
\begin{align*}
\{ & \text{major premise} \\
\{ & \text{minor premise} \implies \text{conclusion.}
\end{align*}
\]

The meaning behind this syllogism is that if the major premise and the minor premise are true then the conclusion is also true.

A simple example on Aristotelean syllogism is
\[
\begin{align*}
\{ & \text{human is mortal} \\
\{ & \text{Sven is human} \implies \text{Sven is mortal.}
\end{align*}
\]

which has an analogue in Theory of Sets:
\[
\begin{align*}
\{ & A \subset B \\
\{ & B \subset C \implies A \subset C.
\end{align*}
\]

Now we give more examples on Aristotelean syllogism and its variation. They are
\[
\begin{align*}
\{ & A = B \\
\{ & B = C \implies A = C,
\end{align*}
\]
\[
\begin{align*}
\{ & A = B \\
\{ & C = D \implies A + C = B + D,
\end{align*}
\]
\[
\begin{align*}
\{ & A = B \\
\{ & C = C \implies A \cdot C = B \cdot C.
\end{align*}
\]

**Example 5.1.1.** In this example we shall determine the derivative of a function \(x^4 \cdot e^{2x}\). If \((f \cdot g)' = f' \cdot g + f \cdot g'\) and if \(f = x^4\), \(g = e^{2x}\); then
\[
(x^4 \cdot e^{2x})' = 4x^3 \cdot e^{2x} + x^4 \cdot e^{2x} \cdot 2 = 2x^3 \cdot e^{2x} \cdot (4 + x).
\]

Solving equations is repeated applications of syllogism.

**Example 5.1.2.** In this example we shall solve an equation \(2x + 4 = 6\).

The first application of syllogism gives
\[
\begin{align*}
\{ & 2x + 4 = 6 \\
\{ & -4 = -4 \implies 2x + 4 - 4 = 6 - 4 \iff 2x = 2.
\end{align*}
\]

And the second application of syllogism gives
\[
\begin{align*}
\{ & 2x = 2 \\
\{ & 1/2 = 1/2 \implies (2x) \cdot (1/2) = 2 \cdot (1/2) \iff x = 1.
\end{align*}
\]
Example 5.1.3. In this example we shall compute an integral $\int x^2 \cdot e^x \, dx$. The first application of syllogism gives
\[
\begin{align*}
\int x^2 \cdot e^x \, dx \\
\int f \cdot g \, dx &= f \cdot G - \int f' \cdot G \, dx
\end{align*}
\]
$$\implies \int x^2 \cdot e^x \, dx = x^2 \cdot e^x - 2 \int x \cdot e^x \, dx$$

where $G' = g$. And the second application of syllogism gives
\[
\begin{align*}
\int x \cdot e^x \, dx \\
\int f \cdot g \, dx &= f \cdot G - \int f' \cdot G \, dx
\end{align*}
\]
$$\implies \int x \cdot e^x \, dx = x \cdot e^x - \int e^x \, dx.$$  

The third application of syllogism gives
\[
\begin{align*}
\int x^2 \cdot e^x \, dx &= x^2 \cdot e^x - 2 \int x \cdot e^x \, dx \\
\int x \cdot e^x \, dx &= x \cdot e^x - \int e^x \, dx = x \cdot e^x - e^x + C_1
\end{align*}
\]
$$\implies \int x^2 \cdot e^x \, dx = e^x \cdot (x^2 - 2x + 2) + C.$$  

The examples show the very standard way of solving equations and computing integrals, they all are applications of syllogism. In fact I can say without any doubt that almost all deductive mathematics in any educational process in one way or other is applications of syllogism.

5.1.2 Body and Mind

Human knowledge and human power come to the same thing,

(Bacon, 2000, Aphorism III).

Although the Aristotellean syllogism is a supreme tool in Aristotellean formal logic, it can not produce new knowledge, for the conclusion (to be asserted) is based on the two premises (to be assumed). Even if the latter is true, the former is still contained in the former, that is the conclusion is always contained in the premises. This was pointed out clearly by Francis Bacon (2000, Aphorism XIII and XIV) and by René Descartes (1984-5, Vol. I, Discourse on the Method). The issue here is not on syllogism itself, but on its role in science, especially the role as scientific method. For Bacon and Descartes, if the method has any scientific value then it must produce new knowledge. Inasmuch as the Aristotellean syllogism can not produce new knowledge, it can not be considered as a general scientific method. Thereby both Bacon (2000) and Descartes (1984-5) made attempt to search for new method and recorded their conclusions therein. Since the method of Descartes is much more relevant to the present essay than that of Bacon, I shall thereby concentrate myself on Descartes. The formulation of Descartes’ method (scientific method) consists of four points to which I quote:
When I was younger, my philosophical studies had included some logic, and my mathematical studies some geometrical analysis and algebra. These three arts or sciences, it seemed, ought to contribute something to my plan. But on further examination I observed with regard to logic that syllogisms and most of its other techniques are of less use for learning things than explaining to others the things one already knows or even, as in the art of Lully, for speaking without judgement about matters of which one is ignorant. · · · For this reason I thought I had to seek some other method comprising the advantages of these subjects but free from their defects. · · ·

The first was never to accept anything as true if I did not have evident knowledge of its truth: that is, carefully to avoid precipitate conclusions and preconceptions, and to include nothing more in my judgements than what presented itself to my mind so clearly and so distinctly that I had no occasion to doubt it.

The second, to divide each of the difficulties I examined into as many parts as possible and as may be required in order to resolve them better.

The third, to direct my thoughts in an orderly manner, by beginning with the simplest and most easily known objects in order to ascend little by little, step by step, to knowledge of the more complex, and by supposing some order even among objects that have no natural order of precedence.

And the last, throughout to make enumerations so complete, and reviews so comprehensive, that I could be sure of leaving nothing out.


The method, formulated so, is not only a general scientific method, but indeed a general didactical method; it should have a far-reaching consequence in mathematics education, especially in the domain of problem solving.

What Bacon and Descartes achieved did not end here, indeed, they engaged in a much bigger enterprise, that is to unite the material world with the abstract world.

As it was said earlier in this chapter, the metaphysical position held by Plato is that things in the material world are projected shadows of Forms in the abstract world, and body is nothing but illusion which could deceive mind, and knowledge can only be found in the abstract world. However, Aristotle, on the other side, argued that knowledge is rooted in the material world and from here he went even further. Indeed, he said of knowledge comes from abstraction of relative relations among material things. Probably there is one common position which was held by both Plato and Aristotle. That is it is through body that we know the material world and it is through mind that we know the abstract world. It could be said that between body and mind Plato emphasizes mind while Aristotle emphasizes body.

I would not claim that Bacon and Descartes succeeded in uniting body and mind in a universally accepted way. However, they did conceive of a more balanced relation between body and mind. Indeed Bacon’s slogan quoted above is simply a balanced relation between body and mind. On the road of uniting body and mind Descartes went much further.

Four works place Descartes permanently in history of philosophy. The works are
The second one and the fourth one overlap heavily. The first one contains twenty one rules conducting mind in searching for truth. For instance, Rule Two characterizes objects on which our query is: “We should attend only to those objects of which our mind seem capable of having certain and indubitable cognition.”. In Discourse and for the sake of rightly conducting one’s reason and seeking the truth in the sciences, Descartes invented his famous method which consists of four rules quoted above. Inasmuch as the method is about human reason and free from defects of syllogism, the method is aiming at seeking new knowledge. This new knowledge is in the first place abstract knowledge, so that Descartes accepted the abstract world as a dominating one. The most important criterion for new knowledge is that it is presented to our mind clearly and distinctly just like clear perceiving and distinct sensing in the material world. A relation between the abstract world and the material world proposed by Descartes is through a unification of mind and body. Body can perceive and sense and see the material world by help of hands and eyes while mind can see the abstract world by help of thought. In the unification of body and mind Descartes only trusts mind. In addition, if a thing presents itself to mind clearly and distinctly then it must be in truth. This is so because of instinct of mind. However Descartes failed to explain scientifically the power of such instinct.

Clearness and distinctness are conditions on truth, but the way to approach truth is through doubting.

For a long time I had observed, as noted above, that in practical life it is sometimes necessary to act upon opinions which one knows to be quite uncertain just as if they were indubitable. But since I now wished to devote myself solely to the search for truth, I thought it necessary to do the very opposite and reject as if absolutely false everything in which I could imagine the least doubt, in order to see if I was left believing anything that was entirely indubitable. Thus, because our senses sometimes deceive us, I decided to suppose that nothing was such as they led us to imagine.


Doubting is thinking in Descartes, in fact in doubting everything including doubting itself, one proves one’s own existence. Thus, the fact that I doubt the fact that I am doubting supplies me with the proof of my existence. In addition, I exist in my doubting, so that I am thinking, therefore I exist. Thinking is proving by dialectic, and thinking is looking from different perspectives, and thinking thereby gives rise to a correspondence between
the material world and the abstract world. I have to emphasize that the last mentioned thinking is not thinking in usual sense, it is thinking in Descartes, namely, it is doubting. In applying this correspondence Descartes in the appendix to *Discourse* created a completely new scientific discipline, a modern name to it is *Analytic Geometry*. Geometry in Descartes starts with construction by ruler and compass, therefore construction in the material world, and the knowledge in construction is in the abstract world.

Any problem in geometry can easily be reduced to such terms that a knowledge of the lengths of certain straight lines is sufficient for its construction.

(Descartes, 1954, p. 2).

In this geometry, a correspondence is consequently established between the material world and the abstract world by measuring lengths of certain straight line segments with ruler and compass. This correspondence is the method of coordinate, and it transforms segments into numbers and interrelations among segments into algebraic equations. Thus, in a plane, a point becomes \((x, y)\), a line becomes

\[ ax + by = c, \]

and a circle becomes

\[ (x - a)^2 + (y - b)^2 = r^2. \]

It seems Descartes indicates that body is the material world while mind is the abstract world. In other words, the material world is an extension of body while the abstract world is an extension of mind. This contention was later picked up by Karl Marx (1954, p. 175).

### 5.1.3 Reason

The title means human reason. It is true that more than one million species live on the earth and it is also true that the human species distinguishes from all others by possessing reason. It is therefore unavoidable and even necessary that reason has been being an utmost and predominant topic in philosophy since the antiquity. Plato, Aristotle, Kant, Hegel, Leibniz, Descartes are among those who made substantial contributions to understanding of reason; and their works have enlarged and transfigured the entire concept, the structure and the methodology of philosophy.

*Reason* as such is immediate on the one side, it can not be determined by a finite sequence of texts for any reasonable determination of reason must lie within reason itself. On the other side reason is susceptible of mediation for it manifests itself in language in which forms of reason are stored and displayed in the first place. The existence of reason preconditions that of self and other, they are subject and object. It is the nature of reason that subject and object can only be connected by reason via mediation. In
order to wake reason up from slumber a subject must act upon an object. The action must be sensible to reason and permitted by the object, this is *knowing*; at the same time there creates a reaction upon the subject, this is *understanding*. Thereby knowing and understanding are two fundamental things that mediate between subject and object. Knowing is toward object

\[
\text{subject} \rightarrow \text{object} \rightarrow \text{Knowing}
\]

whereas understanding is toward subject

\[
\text{object} \rightarrow \text{subject} \rightarrow \text{Understanding}
\]

Within reason the most fundamental constituent of knowing is *intuition* to which corresponds dually *sensibility*, the most fundamental part of understanding. Intuition is toward object

\[
\text{subject} \rightarrow \text{object} \rightarrow \text{Intuition}
\]

whereas sensibility is toward subject

\[
\text{object} \rightarrow \text{subject} \rightarrow \text{Sensibility}
\]

Although reason can not be defined by a finite sequence of text, it is reasonable to claim that reason is a synthesis of knowing and understanding. Judgment is a synthesis of intuition and sensibility, and thereby is the first approximation to reason. Therefore any discussion on reason must begin with intuition, sensibility and judgment.
Intuition as such is the most fundamental constituent of knowing, it enters on stage in the first encounter between subject and object therefore makes a beginning in a process of knowing. Consciousness, unconsciousness, perception, sensation are all related to intuition, of which knowing gets its fuel, by which an object extends into our knowing mind. Action of intuition on an object forms extension of that object, it is this extension that extends into mind, not the object itself, this extension is appearance. Each object having appearance as its immediate extension has other extensions as well, they are mediated extensions. The word appearance has its origin in classical philosophy due to which the meaning of the word is given canonically to geometric appearance. In an era where laws of nature are of quantum theoretical the meaning of this very same word includes stochastic patterns as well.

Sensibility is dual to intuition and thereby anti-intuition and directed towards subject. It is through sensibility a subject gets its very first feeling about an object, this is sensation. All forms of thinking are rooted in a synthesis of intuition and sensation. In fact some of the most significant works of some prominent thinkers are on intuition and sensation, Kant on intuition, Hume on perception, and Mach on sensation, to mention just a few.

Judgment is a synthesis of intuition and sensibility and is the first approximation to reason. Judgement is the foundation of any query conducted by reason, it is embryo upon which scientific theories are to be developed and it is a living organism. Like any other living organisms judgment finds itself in a process of metabolism; it is born, sublated and annihilated. Judgment is a relation between subject and object, therefore judgment can be categorized into two categories. An immediate judgment depends only upon subject and object without referring to others. When I see a book I judge that it is a book, and when I put my hands into cold water I judge that it is cold; they are examples of immediate judgments. However, as a judgment in an elementary school, that of two plus three is five is a mediated judgment, for two, three and five are all mediated by, say, book.

Two central questions in Epistemology (Theory of Knowledge) are the question of what is knowledge and the question of how is knowledge possible. These two questions are not only central questions in epistemology but also two of the central questions in the entire philosophy, consequently were in focus of philosophical discussions in the past thousands of years, and the discussions shall probably be continued for other thousands of years. Each philosopher in the discussions offers his own perspective on the matter and at the same time waits for stormy attacks from the others. Most of perspectives are filled with contradictions at a close examination and are buried in oblivion. A few of the most controversial survive for the time being, they survive not because of being accepted as the correct ones but being so insightful that practical use can be made and more powerful perspectives can
be generated. A close look at all these perspectives shows that a few things are common to all, one of them is reason. Reason is one of the human faculties in the sense of philosophy, not psychology. Reason produces all contradictions thereby is of contradictory, being so, it can not be determined within the realm of philosophy down to the last details. Indeed, the most fundamental determinations of reason must be postulated, as a result, a system or a structure constructed by reason is an axiomatic system or an axiomatic structure.

Among philosophers who made contributions on reason in one way or other, Kant and Hegel are singular. Among Kant’s works the most important one is (Kant, 1998), it consists of the part of Elements (ibid., pp. 151–623) and the part of Method (ibid., pp. 625-704). Kant opens Elements with the following lines:

Auf welche Art und durch welche Mittel sich auch immer eine Erkenntnis auf Gegenstände beziehen mag, so ist doch diejenige, wodurch sie sich auf dieselben unmittelbar bezieht, und worauf alles Denken als Mittel abzweckt, die Anschauung.

(Kant, 1966, A19/B33).

Here I quote the original German text because of the word Erkenntnis. In the English translation which I use consistently throughout, that word has been rendered into cognition whose corresponding adjective cognitive is, in my context, a psychological category, not philosophical one. In addition, Erkenntnis in Hegelian system has two categories – inward and outward – which correspond understanding and knowing respectively, so that Erkenntnis is a synthesis of understanding and knowing. Kant summarizes Erkenntnis, as a structure in human reason, by the following text which is the first sentence of the very last paragraph in Elements:

So fängt denn alle menschliche Erkenntnis mit Anschauungen an, geht von da zu Begriffen, und endigt mit Ideen.

(Kant, 1966, A702/B730).

The fundamental structure of Erkenntnis that Kant set out and upon which Kant builds his system on human reason is thereby

\[
\text{Intuition} \rightarrow \text{Concept} \rightarrow \text{Idea}.\]

Kant expounds pure reason by using that structure which places intuition at a special position — the absolute beginning. Intuition, as the absolute beginning, just like the absolute beginning of logic in Hegelian system — Being, lacks of complete determination except for some partial essence. Here by essence I mean roughly $B$ in an assertion of $A$ is $B$. In fact the essence of intuition given by Kant is intuitive, so that the essence of intuition given by Kant lies in intuition itself. Hence, Kantian definition of intuition is, just
like the definition of point and line given by Euclid (Heath, 1908), an axiom. Insofar as definition of intuition is postulated as an axiom, the fundamental structure of Erkenntnis expressed is axiomatic, consequently susceptible of different interpretations.

Any determination of intuition must have form of

$$\text{intuition is } \ldots$$

where the word \textit{is}, as philosophical category, is the same as the word \textit{being}; and the symbol \ldots is essence. Therefore intuition alone is equivalent to \textit{being} together with \textit{essence}. Stated so, it seems to me very natural to interpret the part of

$$\text{Intuition } \rightarrow \text{ Concept}$$

in Kantian system as a syllogistic one

![Syllogistic Diagram](concept-being-essence)

which was indeed a path that Hegel (1892/1959) chose. We have thereby interpreted the first half of the structure of Erkenntnis by syllogism. This gives rise to a natural question of can the second part

$$\text{Concept } \rightarrow \text{ Idea}$$

be interpreted by syllogism as well. The concept, being as a synthesis of being and essence, is a subjective concept; it becomes an idea only if it is sublated by object. It is therefore reasonable to interpret the mentioned second part in Kantian system by syllogism

![Syllogistic Diagram](idea-concept-object)

which in fact is one of the constituents in the Hegelian system. Hegel thereby carried out a syllogistic analysis on the fundamental structure of Erkenntnis formulated by Kant.

Being in Hegelian system, like intuition in Kantian system, is the absolute beginning and therefore lacks any essence, but it is precisely this lacking of essence that is the very essence of being, consequently the determination of being. The determination of being appeals to a contradiction, to wit, essence of being is that being does not have any essence. The fashion of determination by contradiction is a very important way of determination in
philosophy. In fact, Descartes’ determination of self is of the same character, for when I doubt everything, of course, I also doubt that fact that I doubt, which proves that I exist. Another example is the determination of number zero and number one by Frege to which I shall return later.

Hegel was definitely a master of mastering these seemingly contradictory and unreasonable things, and turned them into a consistent part in reason. However, as to why reason works in this way, Hegel failed to supply with sustainable arguments, rather than appealed to some super-natural power by calling it cunning of reason (die List der Vernunft). Hegel should not be criticized for this, for the essence of cunning of reason determined by him is so important and valuable that the most fundamental doctrines of both Piagetian system and Vygotskian system stem directly from it. Hegel (1892/1959) described the cunning of reason in paragraph 209 with following lines:

Purposive action, with its Means, is still directed outwards, because the End is also not identical with the object, and must consequently first be mediated with it. The Means in its capacity of object stands, in this second premise, in direct relation to the other extreme of the syllogism, namely, the material or objectivity which is pre-supposed. This relation is the sphere of chemism and mechanism, which have now become the servants of the Final Cause, where lies their truth and free notion. Thus the Subjective End, which is the power ruling these processes, in which the objective things wear themselves out on one another, contrives to keep itself free from them, and to preserve itself in them. Doing so, it appears as the Cunning of reason.

(Hegel, 1892/1959).

Hegel then added a remark to the paragraph 209, however this remark does not appear in some German editions of the work. The remark is

Reason is as cunning as it is powerful. Cunning may be said to lie in the inter-mediative action which, while it permits the objects to follow their own bent and act upon one another till they waste away, and does not itself directly interfere in the process, is nevertheless only working out its own aims. With this explanation, Divine Providence may be said to stand to the world and its process in the capacity of absolute cunning. God lets men do as they please with their particular passions and interests; but the result is the accomplishment of—not their plans, but His, and these differ decidedly from the ends primarily sought by those whom He employs.

(Hegel, 1892/1959).

The cunning of reason is an important postulate in Hegelian system, it emphasizes an active, positive, powerful and complex side of reason. Hegel himself returned to it repeatedly even in other works. In the paragraph 54 of Preface to Phenomenology of Spirit, the cunning of reason was given the credit for change of direction of knowing from outward to inward, in other words, it is the cunning of reason that makes understanding (knowing inward) possible. There are some short discussions on the cunning of reason by Hegel in the section on Measure in (Hegel, 2010) and in Introduction to Lectures on the Philosophy of History.
The cunning of reason appears so long as there is action by reason, and man is a unification of body and mind, an extension of body is the material world while an extension of mind is the abstract world. Actions by reason are human processes which include labour process, psychological process, educational process, and so on. As early as 1867 Karl Marx observed an important role played by the cunning of reason in economic processes in terms of labour process.

An instrument of labour is a thing, or a complex of things, which the labourer interposes between himself and the subject of his labour, and which serve as the conductor of his activity. He makes use of the mechanical, physical, and chemical properties of some substances in order to make other substances subservient to his aims. Leaving out of consideration such ready-made means of subsistence as fruits, in gathering which a man’s own limbs serve as the instruments of his labour, the first thing of which the labourer possesses himself is not the subject of labour but its instrument. Thus Nature becomes one of the organs of his activity, one that he annexes to his own bodily organs, adding stature to himself in spite of the Bible. As the earth is his original larder, so too it is his original tool house. It supplies him, for instance, with stones for throwing, grinding, pressing, cutting, etc. The earth itself is an instrument of labour, but when used as such in agriculture implies a whole series of other instruments and a comparatively high development of labour.


The footnote after “his aims” equals to Hegel’s text “Reason is as cunning as ⋅⋅⋅ its own aims”. Marx’ usage of the cunning of reason is indirect but his analysis on tools in labour process is concise and precise. Let me recapitulate it in the framework that has been developed in this chapter. Body, mind and the two worlds are united in the following diagram:

```
man

the material world = body

mind = the abstract world
```

Labour is action by reason and takes place between man and material things, and labour is conducted by tools which are also material things, and thereby any material thing has two sides. An active side of a thing is that the thing is a tool in labour and a passive side is that the thing is a subject upon which labour acts. Thus let \( A \) be a material thing. \( A \) as a tool in labour shall be denoted by

\[ +A \]

which is outward and active, whereas \( A \) as a subject upon which labour acts shall be denoted by

\[ -A \]
which is inward and passive. One of functions of cunning of reason is to interchange $\pm A$. Thus

$$
+A \xrightarrow{\text{cunning of reason}} -A
$$

or

$$
-A \xrightarrow{\text{cunning of reason}} +A.
$$

The material world being a tool house, a labour process has to be taken as a process:

$$
+A \xrightarrow{\text{cunning of reason}} -A \xrightarrow{\text{labour}} -B \xrightarrow{\text{cunning of reason}} +B.
$$

Labour in combination with (cunning of) reason transforms tools into tools, hence labour executed by man in return changes body, and finally changes mind and man himself. Marx’ narrative on changes in man by labour is also one of important factors that forms Vygotskian thought (Vygotsky, 1987-1997, Vol. 4, p. 62) or (Vygotsky, 1978, p. 54)). Marx’ contention is further elaborated byEngels (1940, chapter IX).

Labour is the source of all wealth, the economists assert. It is this—next to nature, which supplies it with the material that it converts into wealth. But it is also infinitely more than this. It is the primary basic condition for all human existence, and this to such an extent that, in a sense, we have to say that labour created man himself.

Thus the hand is not only the organ of labour, it is also the product of labour.

In short, the animal merely uses external nature, and brings about changes in it simply by his presence; man by his changes makes it serve his ends, masters it.

Let us not, however, flatter ourselves overmuch on account of our human conquest over nature. For each such conquest takes its revenge on us.

(Engels, 1940, chapter IX).

A labour process can be succinctly represented as the following scheme in which the fact that labour changes man is emphasized.

$$
\text{man} = U(\text{body, mind}) = U(MW, AW) = U(T, AW)
$$

$$
\xrightarrow{\text{labour}} U(T, AW) = U(MW, AW) = U(\text{body, mind}) = \text{man}
$$

in which $T, MW, AW, U$ stand for tool, the material world, the abstract world, unification respectively. I shall in what follows call this scheme the formula for man in labour process. The formula describes a process of man changes himself by his own labour, and what we did in this section is a syllogistic analysis on this process.
5.2 The Cunning of Reason in Three Contexts

5.2.1 Lev Vygotsky

In contemplation on a question of what is Vygotskian thought, one could take either ontological perspective or practical perspective. As to the former a great deal of progress has been made, thanks to the publication of (Vygotsky, 1986), (Vygotsky, 1978), (Vygotsky, 1987-1997) and (Vygotsky, 1998-1999); as to the latter didactical principles and teaching practices based on Vygotskian thought have existed for quite some time. Influenced by Hegel and Marx, it is not surprising that the fundamental part of Vygotskian thought bears a syllogistic trait. In this section I shall discuss that trait.

The background of Vygotskian thought is the behaviorism in which the fundamental structure is

\[ \text{stimulus} \rightarrow \text{response}. \]

For Vygotsky this structure represents a behavior in lower form, and the structure represents a behavior in higher form is a syllogistic structure showed below (Vygotsky, 1978, p. 40):

\[
\begin{array}{c}
S \\
\downarrow \\
R \\
\downarrow \\
X
\end{array}
\]

This diagram, though a syllogism, lacks of precise definition; thereby the Vygotskian thought can not be read off. In order to see the fundamental idea in that thought, we have to start with Engels's contention on change of man by his own labour. In comparison of Vygotsky with Engels, it is convenient to have the following diagram:

\[
\begin{array}{c}
\text{man} \\
\downarrow \\
\text{mind} \\
\downarrow \\
\text{abstract world} \\
\downarrow \\
\text{sign} \\
\downarrow \\
\text{labour in terms of sign}
\end{array}
\]

\[
\begin{array}{c}
\text{Vygotsky} \\
\downarrow \\
\text{body} \\
\downarrow \\
\text{material world} \\
\downarrow \\
\text{tool} \\
\downarrow \\
\text{labour in terms of tool}
\end{array}
\]

Let me first explain the meanings of some words in the diagram. The word sign, symbol and language in Vygotsky have the same meaning, the word
labour in terms of sign means thinking in terms of sign or simply thinking, and the word labour in terms of tool means simply labour.

The similarity between Engels’ discourse on tool and labour and Vygot-
sky’s discourse on language and thought is too striking to be not noticed. In
fact the similarity is not parallel similarity rather mirror reflection, thereby
Engels’ discourse and Vygotsky’s discourse are dual to each other.

Thus, let A be a sign. The A has an inward part \(-A\) as well as an
outward part \(+A\). As before, one of functions of the cunning of reason is to
interchange \(\pm A\). Thus

\[
+ A \xrightarrow{\text{cunning of reason}} - A
\]

or

\[
- A \xrightarrow{\text{cunning of reason}} + A.
\]

The abstract world being a vocabulary of signs, one of the most fundamental
part of a thinking process has to be postulated as a process:

\[
- A \xrightarrow{\text{cunning of reason}} + A \xrightarrow{\text{thinking}} + B \xrightarrow{\text{cunning of reason}} - B.
\]

The thinking process so postulated emphasizes the action of communication
and thereby has its origin in society. Thinking in combination with reason
transforms inward language into itself, hence thinking executed by man in
return changes man himself. Thinking process, dual to labour process, can
be summarized in the following scheme:

\[
\text{man} = U(\text{body, mind}) = U(MW, AW) = U(MW, -L)
\]

\[
\xrightarrow{\text{thinking}} U(MW, -L) = U(MW, AW) = U(\text{body, mind}) = \text{man}
\]

in which \(-L, MW, AW, U\) stand for inward language, the material world,
the abstract world, unification respectively. I shall in what follows call this
scheme the formula for man in thinking process.

The duality between labour (in terms of ) in Engels and thinking in
terms of sign in Vygotsky is weakly indicated in the following text.

The tool’s function is to serve as the conductor of human influence on the
object of activity; it is externally oriented; · · · The sign, on the other hand,
changes nothing in the object of a psychological operation. It is a means of
internal activity aimed at mastering oneself; the sign is internally oriented.


As to the question of why tool is outward and sign is inward, Vygotsky,
just like Hegel, has no good answer but appeals to the cunning of reason
(Vygotsky, 1978, p. 54). Since the cunning of reason plays such an important
role in labour and thinking, I shall write the formula for man in labour
process and the formula for man in thinking process in another way, so that
the cunning of reason shall be put on a right place. Thus
• the formula for man in labour process becomes

\[
\begin{array}{c}
\text{body} \\
\downarrow \\
\text{tool} \\
\downarrow \\
\text{cunning of reason}
\end{array} \quad \begin{array}{c}
\text{body} \\
\downarrow \\
\text{cunning of reason}
\end{array}
\]

• the formula for man in thinking process becomes

\[
\begin{array}{c}
\text{mind} \\
\downarrow \\
\text{sign} \\
\downarrow \\
\text{cunning of reason}
\end{array} \quad \begin{array}{c}
\text{mind} \\
\downarrow \\
\text{cunning of reason}
\end{array}
\]

If we with Vygotsky call labour (in terms of tool) by tool-using activity and thinking in terms of sign by sign-using activity, then a synthesis of them is a mediated activity which is beyond any reasonable doubt one of the most fundamental constituents of Vygotskian thought. A mediated activity is

\[
\begin{array}{c}
\text{man} \\
\downarrow \\
\text{mediation} \\
\downarrow \\
\text{man}
\end{array}
\]

Vygotsky expounds mediated activities, as foundation of higher psychological processes, within the framework of behaviorism. According to him, the behaviorist scheme \(S \rightarrow R\), without using mediation, is a lower psychological process according to which animals, anthropoids and even children at early age behave. A catalyst in these lower processes is instinct. Behaviors in human, when intervened by tool and sign, change from lower form to higher form. There might be other higher forms, but a special higher form was singled out, this is the form of mediation. A lower form \(S \rightarrow R\) corresponds to inner psychological state which has to be modified or corrected by outer mediation. This correction by outer mediation is a higher form of behavior, it should be taken as the foundation of any higher psychological process. Thus within the framework of behaviorism,

• a lower psychological process is

\[
S \rightarrow R
\]
5.2. THE CUNNING OF REASON IN THREE CONTEXTS

- a higher psychological process is

\[
\begin{array}{c}
\text{S} \\
\downarrow \\
\text{X} \\
\uparrow \\
\text{R}
\end{array}
\]

Whether such a diagram of mediated activity is stable depends on the situation. In a mediated activity of Vygotskian context, the stability is formulated as *to control their behavior from the outside* (Vygotsky, 1978, p. 40). Especially the formulation of two questions relating mediation indicates that Vygotsky had indeed stability as defining characteristic of mediation. The first question is of *how does stimulus transforms into response via mediation*, and the second one is of *how does stimulus recover from response via mediation*.

Labour in Engels has rather definite determination and thereby is susceptible of only narrow interpretations, whereas thinking (in terms of sign) in Vygotsky has much wider category and is susceptible of rich interpretations. In one of explanatory examples used by Vygotsky for explaining the stability, thinking and sign are interpreted as memorizing and knot (tying a knot for memorizing a day). Hence the diagram becomes

![Diagram](man -> man)

\[
\begin{array}{c}
\text{tie} \\
\downarrow \\
\text{knot} \\
\uparrow \\
\text{control}
\end{array}
\]

and another example is on conditioned reflex in which case the diagram is

![Diagram](brain -> brain)

\[
\begin{array}{c}
\text{brain} \\
\downarrow \\
\text{cerebral cortex}
\end{array}
\]

One of the central points in Vygotskian thought is an approach to the mastery of its own behavior via mediation. The ability of mastering self is not in-born but develops gradually in human. Before and at the beginning of intervention by tool and sign, the child’s behavior is outward and is regulated by instinct; and the behavior is of lower form. In Vygotsky’s own words: “Prior to mastering his own behavior, the child begins to master his surroundings with the help of speech.” (Vygotsky, 1978, p. 25). It is only
after the transformation of a lower form into a higher form, speech develops inwardly, which marks the beginning of mediation. This is a process of internalization which is possible because of the cunning of reason. Indeed, internalization is an embodiment of the cunning of reason in the structure of mediation. The internalization transforms outward into inward, thereby interpersonal into intrapersonal. And in the higher form, speech becomes a conductor over activities, in Vygotsky’s word: “importance played by speech in the operation as a whole” (Vygotsky, 1978, p. 26). And he then put internalization into the context of education and wrote:

Every function in the child’s cultural development appears twice: first, on the social level, and later, on the individual level; first, between people (interpsychological), and then inside the child (intrapsychological). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All the higher functions originate as actual relations between human individuals.

(Vygotsky, 1978, p. 57).

In the domain of educational psychology, the internalization in mediated activity was further developed by P. Y. Gal’perin. The internalization is achieved through five levels.

We distinguish five levels of an act: (1) familiarization with the task and its conditions; (2) an act based on material objects, or their material representations or signs; (3) an act based on audible speech without direct support from objects; (4) an act involving external speech to oneself (with output only of the result of each operation); and (5) an act using internal speech. These levels indicate the basic transformation of an act as it becomes mental.

(Gal’perin, 1969, p. 250).

Further discussion on Gal’perin and the five levels within the context of mathematics education was offered by Freudenthal (1991, pp. 138–142).

5.2.2 Jean Piaget

No better words can express Piaget’s philosophical position than those he wrote in 1942:

La logistique est l’axiomatique de la pensée elle-même.

(Piaget, 1942).

He held this position throughout his scientific life and never betrayed it. For him, just like Aristotle (see the text quoted earlier which comes from (Aristotle, 1984, p. 1677)), the ontological reality of things is simply not worth of attention, it is those relative relations among things that matter, of which, formalized logical relations are the foremost important ones. This is the fundamental doctrine of modern axiomatics, and is a basic part of the structural
analysis defined in the preceding chapter. In fact Piaget was constantly in processes of searching those logical relations both quantitatively and qualitatively.

The search began with the important work (Piaget, 1926). A fundamental question in this work is of What are the needs which a child tends to satisfy when he talks (Piaget, 1926, p. 1). Thereby Piaget's research on language and thought does not concentrate on a (psychological) state in which the child is located, but instead of on transition from one state to other. For the child, speech is a mean to escape present situation and to transcend to other situations. Consequently, interrelationships between states are in focus not states themselves. A transition from one state to other, according to Piaget, can be achieved by speech, at least this is true for children at early age. Transition is an important subject in developmental psychology, for it brings joy and satisfaction to the child.

The method which Piaget uses in carrying out the research on language and thought in the child has a clear syllogistic trait, as Claparède wrote in Preface to (Piaget, 1926):

Our author shows us in fact that the child's mind is woven on two different looms, which are as it were placed on above the other. By far the most important during the first years is the work accomplished on the lower plane. This is the work done by the child himself, which attracts to him pell-mell and crystallizes round his wants all that is likely to satisfy these wants. It is the plane of subjectivity, of desire, games, and whims, of the Lustprinzip as Freud would say. The upper plane, on the contrary, is built up little by little by social environment, which presses more and more upon the child as time goes on. It is the plane of objectivity, speech, and logical ideas, in a word the plane of reality. As soon as one overloads it, it bends, creaks and collapses, and the elements of which it is composed fall on to the lower plane, and become mixed up with those that properly belong there. Other pieces remain half-way, suspended between Heaven and Earth.

(Piaget, 1926, p. xii).

To be precise, in searching the function of speech, Piaget asked an interesting question of is it certain that even adults always use language to communicate thoughts (Piaget, 1926, p. 1). In other words: is language always outward? Of course for Piaget language is also inward, which is the internal speech. As to the dialectical relation between the internal speech and the external speech, Piaget wrote:

the individual repeats in relation to himself a form of behavior which he originally adopted only in relation to others. In this case he would talk to himself in order to make himself work, simply because he has formed the habit of talking to others in order to work on them.

(Piaget, 1926, p. 2).

We see here a striking similarity between
• to make himself work (Piaget, 1926, p. 2)
• to master his own behavior (Vygotsky, 1978, p. 73)

In the same fashion as we did with Vygotsky, we can interpret the preceding quotation as a syllogism:

\[
\begin{array}{ccc}
\text{man} & \rightarrow & \text{man} \\
\text{external} & \not\rightarrow & \text{internal} \\
\text{speech} & \not\rightarrow & \end{array}
\]

This kind of syllogism is qualitative and important in itself, but for Piaget this was not enough and he wanted quantitative information about thought in the child. Quantitative information about thought is the logic in Piaget. This logic is an axiomatization of thought which in the end leads to a logico-geometric theory of thoughts. This logico-geometric theory is not in the sense of Euclid, but is akin to that in Felix Klein’s Erlangen program.

A geometry in the Erlangen program is determined by two things. The one is space and the other is a group of transformations on that space. The geometry studies figures and constructions in that space which are invariant under all transformations in a group. In short, a geometry under a group amounts to invariants of that group. An emphasis of a geometry is on interrelationships between geometric constructions not on construction itself, thereby a geometry in the Erlangen program is a structural analysis. A quantitative structural analysis consists in quantitative description on such interrelationships, which, in the Erlangen program, manifests in description by groups.

The most important constituent in any logico-geometric theory is a group which describes interrelationships. Unfortunately, a mathematical group is too fine to fit into Piagetian logico-geometric theory of thoughts; and fortunately, Piaget found a replacement which is called grouping.

Altogether there are nine of them, and they are the preliminary grouping of pure equivalences and the grouping I-VIII. I shall here only mention the former.

The preliminary grouping of pure equivalences is a collection of some well-known properties when mathematicians deal with equalities. In this grouping the whole is a collection of all possible equalities \( A = B \). On this whole a partial multiplication is introduced which means that two equalities in general can not be multiplied together but only some special equalities can be multiplied. To be precise we can only multiply \( A = B \) with \( B = C \) and the partial multiplication depends on this order. Thus

\[(A = B) \cdot (B = C) = (A = C),\]
The partial multiplication is associative. Indeed

\[ ((A = B) \cdot (B = C)) \cdot (C = D) = (A = B) \cdot ((B = C) \cdot (C = D)) \]

because of the both sides are equal to \( A = D \). There are many identities with respect to the partial multiplication. In fact all equalities \( A = A \), \( B = B \) and so on are identities, for

\[ (A = A) \cdot (A = C) = (A = C), \quad (A = B) \cdot (B = B) = (A = B). \]

At last, to each equality \( A = B \) we have

\[ (A = B) \cdot (B = A) = (A = A), \quad (B = A) \cdot (A = B) = (B = B), \]

thereby the equality \( B = A \) is the right inverse of \( A = B \), and \( B = A \) is the left inverse of \( A = B \).

This grouping plays a decisive role in Piaget’s axiomatization of thought in the child, and it appears recurrently in Piaget’s works done after 1940.

### 5.2.3 Gottlob Frege

All discussions on numbers made so far are on operational aspect. Although the ontological aspect occupies a very limited space within mathematics education, which is also true within mathematics itself, it is still an important part of mathematics called *Foundations of Mathematics*. It might be not necessary for experts engaging in computational mathematics to know the story told in this section, but it certainly is to some extent necessary for a mathematics educators and students of mathematics teacher education to have such knowledge, at least some of it. For such knowledge together with the knowledge from the operational aspect will definitely help mathematics educators to make a decision on

- what should be avoided
- what should be taken now
- what should be pushed to the future
- what should be take as the measure in making assessment

in teaching numbers. Before we proceed any further, let me say clearly that numbers in this section mean the numbers with the symbolic names:

\[ 0, 1, 2, 3, 4, \ldots \]

and a discussion on them lies within the foundations of mathematics or even better the philosophy of mathematics.

Let us first listen to what one of the greatest number theorists has to say about numbers.
The elementary theory of numbers should be one of the very best subjects for early mathematical instruction. It demands very little previous knowledge; its subject matter is tangible and familiar; the processes of reasoning which it employs are simple, general and few; and it is unique among the mathematical sciences in its appeal to natural human curiosity. A month’s intelligent instruction in the theory of numbers ought to be twice as instructive, twice as useful, and at least ten times as entertaining as the same amount of “calculus for engineers”.

(Hardy, 1929, p. 818).

Let us, on the other hand, also listen to what one of the founders of foundations of mathematics has to say about numbers,

When we ask someone what the number one is, or what the symbol 1 means, we get as a rule the answer “Why, a thing”. And if we go on to point out that the proposition

“the number one is a thing”

is not a definition, because it has the definite article one one side and the indefinite on the other, or that it only assigns the number one to the class of things, without stating which thing it is, then we shall very likely be invited to select something for ourselves—anything we please—to call one. Yet if everyone had the right to understand by this name whatever he pleased, then the same proposition about one would mean different things for different people,—such propositions would have no common content.

(Frege, 1953, p. xiv).

The contrast between the information behind the two texts is too apparent not be noticed. On the one hand, we are led to a picturesque and lyric garden of the kingdom of numbers; while on the other hand, we are showed its melancholic backyard filled with logical contradictions, even the familiar words a and the contribute a paradox.

This is simply a state of affair. Numbers, like most of objects in philosophy, have two categories (aspects). The first one is doing and the second is being. These philosophical categories appear in other disciplines under some other names. In education they become concrete and abstract, in psychology operational and genetic, and in mathematics they become operational numbers and ontological (abstract) numbers.

As the text of Frege told us, the fundamental question about a number say the number two is of how is it possible that two different things, two and two books, mean the same thing. I would put this question in a syllogistic fashion, that is of what is the becoming of two on the one side and two books on the other side. An inquiry into the mentioned becoming can proceed in different paths. One of them is via axiomatics in which some expressions shall be taken as granted. Let us proceed along this path a bit further.

Inasmuch as an axiomatic path is chosen, some expression must be taken as granted. That expression is reified number. In using a reified number, I postulate
• a context $A$ must be prescribed

• the syntax is

$$\text{the number of objects in } A \text{ is } \cdots$$

where $\cdots$ is the reified number reified in $A$.

With this definition, an abstract number is thus a reified number reified in itself, whereas a concrete number is a reified number reified in others. Take for instance a context $A$ as 2, then

$$\text{the number of units in } 2 \text{ is } 2,$$

so that the number 2 becomes the reified number reified in itself. Now take a context $A$ as Brouwer and Hilbert, then

$$\text{the number of mathematicians in } A \text{ is } 2,$$

so that the number 2 becomes the reified number reified in others.

Now we turn to Frege, for any philosophical discussion on numbers cannot be complete without a discussion on the contribution made by Gottlob Frege. His grand ambition is to put mathematics, indeed the whole science, into a logic, Frege’s logic. The attempt was made not only by Frege but also by Peano nearly at the same time. For both the single foundation of the whole mathematics edifice is the so-called mathematical induction, thereby it sufficient the latter into the logic. This amounts to putting the ontological number 1 and the operational number 1 into the logic. Thereby Frege had to inquire into the question of what is the number 1 and the question of what is the addition by 1.

The logic which Frege searches for (or creates while he is searching for) has an absolute center, concept (Begriff). Frege’s logic was developed by him in three steps. The first step is the publication of a booklet Begriffsschrift in 1879 (for English translation see Heijenoort, 1967), in which a symbolic language was created with an aim that all concepts are going to be recorded and stored in that language. The language turned out to be too odd to gain its popularity, therefore failed to be accepted generally. The second step is the publication of another book (Frege, 1953) in 1884, in which concept of number was discussed from critical and philosophical perspective. The third step is the publication of a two-volume book Grundgesetze der Arithmetik in 1893/1903 which shall not be cited any more, for I mention them only here. Of all Frege’s writings, I shall concentrate on (Frege, 1953). This inspired and inspiring work has so far not been examined within the domain of mathematics education, due probably to its mathematical and philosophical character. Grand ambition of the work is to conduct an inquiry into the definition of number and the relation between a number and its immediate successor, the relation where counting is based on. A work of this sort bears
certain resemblance to the system of Peano in one way or another. Except for difficult reading of the book, I find it highly relevant to contemporary mathematics education, especially to the domain of mathematics in context. The fundamental philosophy of Frege in this book is expressed as

\[ \text{die Zahlangabe eine Aussage von einem Begriff enthalte} \]

which is translated as

the content of a statement of number is an assertion about a concept.

Frege’s use of the word concept is very different from the contemporary use of the word, for instance, moon of Venus is a concept while horse that draws the King’s carriage is another.

This extremely important contention appeared in the middle of the book (Frege, 1953). First of all, Frege spent 57 pages of total 119 of that book on a critical examination on traditional concepts of numbers prior to him and summarized it in nine points.

- Number is not abstracted from things in the way that color, weight and hardness are, nor is it a property of things in the sense that they are. But when we make a statement of number, what is that of which we assert something? This question remained unanswered.

- Number is not anything physical, but nor is it anything subjective (an idea).

- Number does not result from the annexing of thing to thing. It makes no difference even if we assign a fresh name after each act of annexation.

- The terms “multitude”, “set” and “plurality” are unsuitable, owing to their vagueness, for use in defining number.

- In considering the terms one and unit, we left unanswered the question: How are we to curb the arbitrariness of our ways of regarding things, which threatens to obliterate every distinction between one and many?

- Being isolated, being undivided, being incapable of dissection—none of these can serve as a criterion for what we express by the word “one”.

- If we call the things to be numbered units, then the assertion that units are identical is, if made without qualification, false. That they are identical in this respect or that is true enough but of no interest. It is actually necessary that the things to be numbered should be different if number is to get beyond 1.
• We were thus forced, it seemed, to ascribe to units two contradictory qualities, namely identity and distinguishability.

• A distinction must be drawn between one and unit. The word “one”, as the proper name of an object of mathematical study, does not admit of a plural. Consequently, it is nonsense to make numbers result from the putting together of ones. The plus symbol in \(1 + 1 = 2\) cannot mean such a putting together.

The first four points tell about what a number ought not to be, of course, this leaves a room for Frege to give his sense of definition in a later part of the book. The last five points carry out a discussion on a distinction between one and unit.

Frege’s definition of number begins with an answer to the question posed in the first point of his nine points of critics. Let us repeat the question, it is of *when we make a statement of number, what is that of which we assert something*. The answer given by Frege is this: the *content of a statement of number is an assertion about a concept*. This principle of Frege is no doubt the most important and the most fundamental one of the entire book (Frege, 1953). When we make a statement on a number, whether we are aware or not, whether we want or not, whether we wish or not, something else is asserted by the cunning of reason. It gives thereby the cunning of reason a decisive role in the conception of number. We record the syllogism as follows

\[
\text{man} \quad \text{assert concept to} \quad \text{man}
\]

\[
\text{state about} \quad \text{man} \quad \text{number}
\]

The principle puts statement about number and concept of number on an equal footing and at the same time combines them in a dialectical fashion. Indeed, in order to define a number I need a concept, in order to have a concept I have to make a statement about a number. The philosophical thought of this principle is the same as that of Hegel’s *The numbers produced by counting are counted in turn* and that of Descartes’ *I doubt everything including that I doubt therefore I exist*. Let us take look at two examples about this dialectical combination given by Frege himself.

If I say “Venus has 0 moons”, there simply does not exist any moon or agglomeration of moons for anything to be asserted of; but what happens is that a property is assigned to the concept “moon of Venus”, namely that of including nothing under it. If I say “the King’s carriage is drawn by four horses”, then I assign the number four to the concept “horse that draws the King’s carriage”.

(Frege, 1953, p. 59).
Some judgments such as that of the number one is a thing and that of two is two books do not contain a concept proper, for they do not supply with contexts. However, if I say that the number one is the thing in my bag, then I assign the number one to a concept of the thing in my bag. Similarly, if I say that the number two is the two books on my desk, then I assign the number two to a concept of the books on my desk. Putting other way around, the number two and the concept of book are united in that judgment. This is what Frege called: “Zahl in Zusammenhange eines Urteils” (number in the context of a judgment).

Now that judgments are in the foundation of numbers, Frege continues with his definition of numbers by considering two forms of judgments. The first one is a judgment about relations between a concept \( F \) and an object \( a \). When \( a \) and \( F \) are given, I have to judge if that object \( a \) belongs to the concept \( F \), if so, the concept gives rise to a number \( n \). Frege expresses as

- an object \( a \) falls under (fallen unter) a concept \( F \),
- a number \( n \) belongs to (kommen zu) a concept \( F \).

Symbolically it is

\[ a \xrightarrow{\text{fall under}} F \xrightarrow{\text{belong to}} n. \]

After these preparations, Frege undertakes his long journey of defining natural numbers in the fashion of mathematical induction.

First of all, Frege began with a definition of the number 0. To this end a suitable judgment is to be found in such a manner that a concept can be extracted and no object can fall under that concept. There are many ways to make such judgment, for instance, a pen sitting on my desk is not identical with itself. This judgment gives rise to a concept of not identical with itself. Since there is no object falling under this concept, the number belonging to it is the number 0 (Frege, 1953, p. 87).

In the spirit of mathematical induction, let us assume that the number \( n \) is already defined. We need now to define the number \( n + 1 \). Suppose that we have a concept \( F \) and an object \( a \) falling under \( F \). We now construct another concept \( G \): “falling under \( F \) but not \( a \)”. If we can show that the number \( n \) belongs to the concept \( G \) then we say that the number belonging to \( F \) is the number \( n + 1 \) (Frege, 1953, p. 67).

With these two pieces in hands, Frege defines the number 1 in the following fashion. Let \( F \) be the concept: “identical with the number 0” (Frege, 1953, p. 90). It is clear that the number 0 falls under this \( F \). Let \( G \) be the concept: “falling under \( F \) but not the number 0” which amounts to “identical with the number 0 but not the number 0”. Since no object can fall under \( G \), the number 0 must belong to \( G \), so that the number belonging to \( F \) is the next number after the number 0. Frege defines the number belonging to the preceding \( F \) as the number 1. Let us put the concepts to which belong the number 0 and the number 1 at one place:
5.3. AXIOMATIZATION OF THE CUNNING OF REASON

• not identical with itself (to which 0 belongs),
• identical with the number 0 (to which 1 belongs).

Here we can see a magnificent work done by the cunning of reason.

In Frege’s perspective a number goes side by side with a concept, each number belongs to a concept, and each concept is located in a judgment. Therefore it is meaningless to talk about a number without specify a judgment which supplies us with a context. Although this can not be considered scientifically satisfactory, its implication to mathematics education is indeed far-reaching. It gives numbers in context, treating as a didactical principle, a philosophical and methodological foundation.

Two attempts of placing mathematics on a solid foundation made by Frege and Peano ended fruitlessly. However the symbolic language invented by Peano gained wide acceptance because of Russell and Whitehead adapted it into the monumental \textit{Principia Mathematica}. Modern way of writing mathematics is strongly influenced by that language. On the other side, the symbolic language of Frege is so special that it never was looked upon seriously by the mathematical community. In addition one of the axioms in Frege’s system gives rise to an inconsistency which was pointed out by Russell. Nonetheless, Frege has to be considered as one of the greatest logicians, for he was the first one to break the dominance of Aristotle.

### 5.3 Axiomatization of the Cunning of Reason

Two principal objects in the present essay are the syllogistic analysis and the cunning of reason. The former concerns itself interrelationships among syllogisms, while the latter consists in syllogism itself. In particular, the cunning of reason is part of a unitary syllogism.

A binary syllogism, in contrast to a unitary syllogism, is relatively manageable by human reason, for it often lies outside of reason. On the other side, a unitary syllogism lies inside reason itself. Action of reason on any unitary syllogism amounts to action of reason upon itself.

In a binary syllogism, the first two sides are always prescribed on an equal footing. A mathematical equation $a = b$ is a typical binary syllogism, for $a$ and $b$ are prescribed sides. A water molecule is also a binary syllogism, for it is a synthesis of one oxygen atom and two hydrogen atoms. An one-to-one correspondence between a set of five books and a set of five apples is another example of binary syllogism. All mathematical structures in Bourbaki are binary.

One of the two typical examples on unitary syllogism is

$$A \rightarrow \text{de-}A \rightarrow \text{re-}A$$
where the re-\(A\) is in fact the sublated \(A\). The other is

\[
\text{when one counts one is counted.}
\]

A character that makes a unitary syllogism unique is the asymmetry between its first side and its second side. The first side is an action of reason, whereas the second side is an action of the cunning of reason.

There seems to exist two kinds of thinking (thought) in reason, the one manifests itself in unitary syllogisms, while the other in binary syllogisms. In addition, these two kinds of thinking seem to be irreducible to each other. If that is the case, then parallel to this, reason is decomposed into two irreducible parts — the unitary reason (the cunning of reason) and the binary reason. The former is responsible, among others, for the inductive thinking, while the latter for the deductive thinking.

An attempt shall be made here to axiomatize the cunning of reason so as to a system shall be found to which some well-known systems are subservient. This axiomatization is not in the sense of mathematics, and a purpose of it is to list out some most common properties that consist in some well-known cases.

A central point of cunning of reason is an action by reason which, in concrete, includes labour, learning, teaching, psychological process, action of force in the sense of physics, and so on. I shall call all these by a single name operator and shall write it as \(T\). In order that an operator can take place, it must first of all exist a thing upon which the operator can act, and then at the end another thing is being produced. All these things are called states of that operator. If \(T\) is an operator which acts upon \(A\) and produces \(B\); then the state \(A\) is called the initial state of \(T\) and the state \(B\) is called the final state of \(T\). I shall write this, in symbols, \(T : A \rightarrow B\) or else

\[
\begin{align*}
A & \xrightarrow{T} B.
\end{align*}
\]

The operator \(T : A \rightarrow B\) enters into the cunning of reason via its primary form.

Let \(\mathcal{A}\) be a category in which \(A\) varies and \(\mathcal{B}\) be a category in which \(B\) varies. If for every \(A\) in \(\mathcal{A}\) it holds that \(T(A) = B\) is in \(\mathcal{B}\), then we write

\[
\mathcal{A} \xrightarrow{T} \mathcal{B}
\]

which is called primary form of operator \(T : A \rightarrow B\).

We shall call an ordered triple \((\mathcal{A}, \mathcal{B}, T)\) a model for the cunning of reason. And we list four axioms on the triple.

- The first axiom postulates that it is not the operator \(T : A \rightarrow B\) but its primary form \(T : \mathcal{A} \rightarrow \mathcal{B}\) that has meaning in reason.
- The second axiom postulates that the primary form of any meaningful operator can not leave a category functionally invariant.
• The third axiom postulates that, for a given operator $T : A \to B$, there exists another operator

$$T^* : B \to C$$

so as to a successive performance of $T$ and $T^*$ (in that order) is meaningful in reason and is to be considered as yet another operator

$$T^*T : A \to C.$$

• The fourth axiom postulates that an initial states and a final states of $T^*T$ are permitted varying in the same category.

Before we proceed any further, let us make some comments on the system of axioms. The second axiom states that if $T : A \to B$ is meaningful then $A$ and $B$ must be functionally different categories. In particular it follows that

$$A \xrightarrow{T} A$$

is impossible for any operator $T$ and any category $A$. I shall call this axiom, in conformity with quantum statistics of similar particles, \textit{principle of indistinguishability}.

In the third axiom, the operator $T^* : B \to C$ is called \textit{adjoint} operator of $T : A \to B$. Let $T : A \to B$ and $T^* : B \to C$ be the primary forms of the corresponding operators. We call the $T^*T : A \to C$ \textit{secondary} form of the operator $T : A \to B$. The synthesis $T^*T$ of $T$ and $T^*$ can be expressed in the following commutative diagram:

$$\begin{align*}
A & \xrightarrow{T^*T} C \\
& \searrow^{T^*} \quad \swarrow_{T}
\end{align*}$$

The operator $T^*T : A \to C$ is a mediated operator whose form is higher in comparison with the primary form of operator.

The most important is probably the fourth axiom which permits the category of initial states and that of final states to be equal. Thus it is perfectly legitimate to have a commutative diagram

$$\begin{align*}
A & \xrightarrow{T^*T} A \\
& \searrow^{T^*} \quad \swarrow_{T}
\end{align*}$$
in which an initial state $A$ and a final state $T^*T(A)$ are in the same category, but this time $T^*T(A)$ is the higher form of $A$.

In the rest of this section, we shall show some examples which are subservient to the system of four axioms.

The first example is the labour process in Marx and Engels. Let $A$ be the category of tools and let $B$ be the category of material things in nature. Let $T$ stand for labour. A tool, just like hand, is a bodily extension. Let $+A$ be a tool. At the end of a labour process, a material thing $-B$ is produced. That is

$$T : +A \rightarrow -B.$$  

The adjoint operator $T^*$ changes the material thing $-B$ into a tool $+B$, so that

$$T^* : -B \rightarrow +B.$$  

We shall exemplify this by taking $+A$ as hand. The hand produces in labour process a wrench as material thing. It is the cunning of reason that transforms the wrench as material thing into the wrench as tool. Thereby $T^*T$ transforms the hand into the hand holding the wrench, which is a further extension of hand or what amounts to the same thing as a higher form of hand. We show this particular example as follows:

Thereby, when man changes the nature man is changed.

The second example is the thinking process in Vygotsky. Let $A$ be the category of inward signs and let $B$ be the category of outward signs. Let $T$ stand for thinking. Let $-A$ be an inward sign (a piece of internal language). At the end of a thinking process, an outward sign $+B$ (a piece of communicative language) is produced. That is

$$T : -A \rightarrow +B.$$  

The adjoint operator $T^*$ changes the outward sign $+B$ into an inward sign $-B$, so that

$$T^* : +B \rightarrow -B.$$  

Take $-A$ for instance as the idea of days. In order to remember days and at the end of a thinking process, a stick with say seven notches on is produced. It is the cunning of reason that transforms a stick with seven notches on
into an idea of seven days:

\[
\begin{array}{ccc}
\text{days} & T^*T & \text{seven days} \\
\downarrow T & & \downarrow T^* \\
\text{seven notches}
\end{array}
\]

Thereby, when man controls days man is controlled, for man begins to have the sense of week.

The third example is the group structure in mathematics. An underpinning of the structure is the multiplication written as \( a \cdot b \) which is called a product of elements \( a \) and \( b \). Let \( A \) be the category of elements and let \( B \) be the category of products. Let \( T \) stand for multiplication. Then \( T^* \) is the division. Let for instance \( T \) be the multiplication by \( b \) and \( T^* \) is the multiplication by \( b^{-1} \) (the division by \( b \)). In diagram we have

\[
\begin{array}{ccc}
a & T^*T & a = (a \cdot b) \cdot b^{-1} \\
\downarrow T & & \downarrow T^* \\
a \cdot b
\end{array}
\]

The fourth example is from calculus in mathematics. Let \( A \) be the category of functions and let \( B \) be the category of derivatives. Let \( T \) stand for differentiation. Then \( T^* \) is the integration. For instance if a function is \( \sin 5x \). Then we have the following diagram

\[
\begin{array}{ccc}
\cos 5x & T^*T & \cos 5x + C \\
\downarrow T & & \downarrow T^* \\
-5 \cdot \sin 5x
\end{array}
\]

The fifth example is from mechanics. Let \( A \) be the category of active and let \( B \) be the category of passive. Let \( T \) stand for action. Then \( T^* \) is the corresponding reaction.

The sixth example concerns Piaget’s works. In fact, he used intensively the model for the cunning of reason throughout his academic life. Here I mention some of them:
models for the cunning of reason in Piaget

<table>
<thead>
<tr>
<th>$T$</th>
<th>$T^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>sensory</td>
<td>motor</td>
</tr>
<tr>
<td>assimilation</td>
<td>accommodation</td>
</tr>
<tr>
<td>equilibration</td>
<td>de-equilibration</td>
</tr>
<tr>
<td>internal speaking</td>
<td>external speaking</td>
</tr>
</tbody>
</table>

There are many examples of the model which are used in mathematics education. We give here two of them. The first one is that of $T$ is to count in which case $T^*$ is to be counted. The other is that of $T$ is to measure in which case $T^*$ is to be measured.
Chapter 6

Numbers

I had my own kaleidoscope when I was a kid, and liked it very much. Each time when I shook that very same thing a bit and then looked through a peephole, I saw a beautiful pattern. The patterns are so complicated that I could never have mental copies of them down to the last details even after hundreds of times of peeping. Yet they are so regular that I could recognize them immediately when I see them. Each pattern has its own characters which are clear and at the same time distinct from others. Just as I thought of having a complete picture of them, I found something unexpected at the next peeping. These unexpected things can be new local details or new global structures of the old details. To the end I realized that it is probably not meant for me to have complete pictures, for otherwise I might find my kaleidoscope predictable thereby lose interest for it. A great value of a kaleidoscope is, at least for me, that it supplies me with an ever going process of peeping; even if at some particular peeping I find nothing new, I am always confident that I shall find something interesting if I just keep going.

Numbers carry structures. We know numbers through structures. Structures evolve and our knowledge on numbers develops. Structures come and go, but numbers stay. In this chapter I shall show the way that knowledge on numbers can evolve by putting numbers in some known structures. Many syllogistic analyses are carried out in this chapter, but the principal syllogism that guides the whole chapter is

\[ \text{structure} \rightarrow \text{de-structure} \rightarrow \text{re-structure}. \]

An equation \( a = b \) is a syllogism, for it is a synthesis of a side \( a \) and a side \( b \). A syllogistic analysis on equations is thereby interrelationships among them. Hence when I carry out a syllogistic analysis in the present chapter, I shall not mention the name explicitly.

The first section summarizes the discussions made at times in this essay on natural numbers. The set of all natural numbers equipped with the
addition is the foremost important structure upon which the whole mathematics is built. Mathematical emphasis is on one-to-one correspondence, while intuitive emphasis is on counting. Mathematics education on natural numbers is thereby a becoming of these two emphases, and can be seen from at least three perspectives.

The second section conducts a short discussion on multiplication table, and look upon it from the counting perspective. In passing, an alternative to the multiplicative algorithm is given.

The third section begins with an examination of the role played by the number zero in a counting situation. A syllogism in the section leads to the deprivation of the special role played by the number zero. As a result, counting can begin at any number, and the subtraction and the negative integers find their places in the counting perspective. This leads to a well-known construction of all integers and the ring $\mathbb{Z}$.

The fourth section is parallel to the third section in the sense that the addition is replaced by the multiplication. The additive construction has a natural analogue in multiplication, the ring $\mathbb{Z}$ of integers is replaced by the field $\mathbb{Q}$ of fractions. A syllogistic analysis is carried out on the operations of fractions, and some intuitive perspectives are introduced into fractions.

The fifth section begins with a special local structure $1 + 2 + \cdots + n$ and then shows a progressive development of it led by a syllogistic analysis.

The sixth section offers a comprehensive discussion on the method of mathematical induction. The discussion is made within the framework of this section via a minimum structure. The method of mathematical induction is important in mathematics and is challenging in mathematics education. In nearly all modern textbook, only one of many variants of the method is picked up. Since we choose the underpinning of the method to be the minimum structure of natural numbers, we then can treat all variants on an equal footing. Altogether four variants are presented in detail. The material in this section is important and difficult to find elsewhere.

The seventh section aims at giving some quantitative determination of the statement of *God made the integers and man made the rest* said by Kronecker. The aim is achieved by analyzing a book written by Edmund Landau. The analysis is helpful in mathematics education on integers, for it helps mathematics educators in deciding which should be taught now and which should be taught in the future. In addition, knowledge of the book prevents us from falling into a vicious circle.

### 6.1 Natural Numbers

Mathematics is built around numbers, without numbers without mathematics. In some natural languages mathematics simply means numbers and counting. It is through counting, number in mind and number in body are
united. Number in body manifests itself in objects which are presented to mind via intuition, and objects in this respect are bodily extensions. Body extends itself at will or at random; in counting at early age the randomness dominates. Number in mind is of necessity, so that number in an educational process is in the becoming of necessity on the one side and contingency on the other side. In mathematics education contingency leads to concreteness while necessity to abstractness.

Numbers in this section mean natural numbers, those numbers with the symbolic names 1, 2, 3, \ldots. Some even call the number zero a natural number, but this is not universally agreed upon. Whether 0 is a natural number or not, it is a matter of taste or convenience. Nevertheless the collection of all natural numbers is no more and no less than what is needed to build the entire mathematics edifice.

The science of natural numbers is one of the oldest branches of mathematics, it concerns itself interrelations among numbers. Interesting relations can be formulated after natural numbers are further classified. It turns out that the classifications depend on the ways that numbers group together. A typical grouping is to arrange the number of things, say fattened black points \( \bullet \) which are called bullets, in a rectangular matrix:

\[
\begin{array}{cccc}
\bullet & \bullet & \cdots & \bullet \\
\bullet & \bullet & \cdots & \bullet \\
\vdots & \vdots & \ddots & \vdots \\
\bullet & \bullet & \cdots & \bullet \\
\end{array}
\]

and matrix as such is assumed to have more than one row and more than one column. A matrix with only one row or column shall be called a row or a column. If things can be arranged into a matrix with exact two rows or two columns then the corresponding number is an even number. Possibility of arrangement into a matrix gives rise to multiplication and divisibility while impossibility to prime numbers. Visual grouping appeals to intuition, and a specially formalized grouping has been being one of very central topics in natural numbers since the antiquity. A topic is on structure of solutions in natural numbers of some Diophantine equations of which \( ax + by = c \), \( x^2 - ny^2 = 1 \) (Pell's equation), \( x^n + y^n = z^n \) appear regularly.

The process of knowing natural numbers begins with bodily extensions which have the same category of bodily organs, and consequently the beginning is psychological. It is necessary that the subject is aware of self and in addition the self is in the center of knowing and extends itself at random. At this stage the most important cognitive states are 5 \( \rightarrow \) five books, 2 \( \rightarrow \) two apples, and so on. In knowing 2, not only 2 \( \rightarrow \) two apples but also 2 \( \rightarrow \) two books, 2 \( \rightarrow \) two stars and similar states must be taken into consideration. In fact all these states must be treated on an equal footing.
The mathematical ontology of the number 2 is the commutative diagram:

\[ \begin{array}{c}
2 \\
\rightarrow \\
| \\
\rightarrow \\
| \\
\rightarrow \\
I \\
| \\
\rightarrow \\
| \\
\rightarrow \\
II \\
| \\
\rightarrow \\
| \\
\rightarrow \\
III \\
\rightarrow \\
\text{two books} \\
\end{array} \]

in which the third side is an one-to-one correspondence between a set of two books and a set of two apples.

Natural numbers in mathematics education are in the becoming of one-to-one correspondence on the one side and counting on the other side. Counting leads naturally to addition which supplies the set of all natural numbers with a principal structure.

Everyone knows some numbers and has some favorite numbers and dislikes some numbers, even the ones who claim that they hate mathematics. Any teacher needs not worry about how children feel about small numbers such as 1, 2, 3, 4, 5. The children have good feelings about those numbers to which things can be associated by them. For instance, the number 3 is associated to three apples and the number 5 to five candies. They probably can even tell you without thinking much which of those numbers is the biggest and which is the smallest and which are in the middle. This kind of talent in children develops automatically in a normal social surroundings. All these facts must be postulated as axioms in an educational process.

Once I asked a girl of five years old if she can give me a number and she gave me the number 2; and I asked her if she knew the meaning of that number 2, she replied with: “2 means 2 books”. Some days later I met the same girl again and asked her the same two questions. But this time she gave me the number 2 and “2 means 2 balls”. The two answers contain no surprise and are what one can expect in a normal situation. In the first place the number 2 is the name to something (favorite imaginary friend for instance) the child is familiar with. In the second place the child plays with this friend via its bodily extensions manifesting in books or balls. This is the perspective of play, and this perspective is intimately related to the statistical perspective.

We live, from the statistical perspective, in a contingent world and things are given to us at random. In order to organize our lives we must possess certain abilities to make decisions based on random phenomena. One of the abilities is to produce mental copies of a real object and to make judgments on the copies. This particular ability is the fundamental force that balances between contingency and necessity. Every time we see a phenomenon related to a number, we see a contingent medium (book or apple) which carries the number. Through making mental copies of the phenomenon of that number, the real objects are homogenized, and the homogenized objects
are imaginary objects in mind. The two homogenized apples are copies of each other, though the two real apples have different shapes and weights. Homogenized objects are in an important middle stage in the process of abstracting the number 2 from a set of two apples.

The number 2 treating as a mathematical object is not the same as two books or two balls or any other two things. Although the number 2 and two books are not the same thing, but they are closely related, so closely that one can not exist without the other. Let us look upon two books as a phenomenon of the number 2. Thereby the number 2 behaves much like a stochastic variable in which case values of it are phenomena. Thus the number 2 is a stochastic variable and a set of two books is a observation or a sample or a reification of that variable.

Besides the perspective of play and the statistical perspective; there is the dialectical perspective in which the number 2 is a thesis while a set of two books is an antithesis. Let us see what happens when the thesis and the antithesis have passed into each other. When the number 2 passed into the set of two books, it results in

the number 2 is in the set of two books,

thereby the number 2 is reified in the set of two books. Whereas the other way around results in

the set of two books is in the number 2,

so that the set of two books is reified in the number 2, this can also be expressed as that the set of two books is de-reified. De-reification is nothing but abstraction while de-abstraction is reification.

Abstraction and reification often refer to processes which seldom have well-defined ends. If the number 2 is not well-defined then the process of abstracting the number 2 out of a set of two books does not make a rigorous sense. If on the other side the number 2 is well-defined then the genuine meaning of abstraction of the number 2 out of a set of two books is that the set of two books is reified in the number 2. In mathematics education a principal aim is not to define the number 2 but to describe interrelationships between the number 2 and a set of two books, and the description should be made from all three above-mentioned perspectives.

In general, a discussion about processes without well-defined end is philosophical and in particular, a discussion about abstraction without well-defined end lies at the foundation of mathematics. Abstraction should not be used as a catchword if it lacks of determination, and the use of the word should be kept minimal unless the determination is substantial. However some aspects of abstraction are in fact valuable in mathematics education, one of them is the ergodic aspect.
The word **ergodic** determines a category in statistical mechanics, and the full use of its determination is not made here, instead of the syllogistic aspect of the word is relevant here. **Ergodic** is a synthesis of the category **distinguishable** and the category **indistinguishable**, that is

\[
\text{ergodic} \quad \begin{tikzpicture}
\node (ergodic) {ergodic};
\node (distinguishable) [below left of=ergodic] {distinguishable};
\node (indistinguishable) [below right of=ergodic] {indistinguishable};
\path (ergodic) edge (distinguishable)
(distinguishable) edge (indistinguishable)
(ergodic) edge (indistinguishable);
\end{tikzpicture}
\]

Existing material things cannot be indistinguishable, for material things that exist must be in motion and thereby two material things must exist at different space-time. We can always distinguish material things by their locations in space-time. However, in mind things are independent of space-time, therefore there exist distinct things that are in fact indistinguishable. These distinct things are mental copies of another thing. Thus five distinguishable material books become five indistinguishable yet distinct things in mind. This is the ergodic aspect of the category of distinguishable and the category of indistinguishable, and this aspect is an important constituent of abstraction. I believe that the ergodic aspect of abstraction is much more relevant than abstraction itself in mathematics education. Producing small number of mental copies of one object is so natural that even children can do that after some suitable training, in fact this activity is repeated so many times that it becomes a subconscious activity.

### 6.2 Counting

Arithmetic begins with counting, and counting differs from calculating and computing. The latter emphasize the end results obtained by whatever means, whereas counting refers to that special means which we are used to deal with small numbers since our childhood. Counting then leads to the addition which further leads to the multiplication.

However these two operations are taught and learned monotonically in schools, so much that most of pupils appeal to calculators when they add and especially when they multiply. A few use some ready-made algorithms at times. Before the multiplication table is introduced, the addition and the subtraction make practical sense for most of children. At the turn of the multiplication table the intuitive counting becomes a piece of formalized computation by making use of ready-made algorithms. Nearly all algorithms in school context are built upon the multiplication table which each child must learn by rote, as if the whole mathematics edifice is constructed on that table.

In theory the multiplication table facilitates counting for those who have learned counting, but in practice it yokes mathematical behavior for those in
learning. In modern schools the multiplication table has become a totalization of counting. A universal and efficient way to counteract a totalization is to find a dialectic, meaning to be able to see the multiplication table from other perspectives.

I still remember the excitement the first time when I encountered with the multiplication table, it makes long computations shorter. I also remember that later the excitement so decayed that I experienced boringness. Fortunately, it was compulsory to use not only the multiplication table but also an abacus in China at time when I was in elementary school. I found it more interesting to alternate the multiplication table with an abacus.

It seems that I am not alone to this experience. In the past years I taught at times courses on didactics to the students in mathematics teacher education as well as students in mathematics in-service teacher education; and each time when I discussed the multiplication table, I also picked up a variation of abacus. The result is positive so that now I present some parts that I taught in those courses.

Of course I could not use an abacus directly, but instead, I used the principle that rules over abacus to find a dialectic to the multiplication table. In this way, I put the formalized multiplication back into the intuitive counting.

In order to do this, let us mark the number 1 with a stick | and the number 10 with a fattened stick and so on. Thus the number 235 can be represented as:

\[
\begin{array}{c}
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
2 \cdot 100 + 3 \cdot 10 + 5 \cdot 1 \\
\end{array}
\]

235

As usual, we arrange things in a row or in a column when we add and arrange things in a rectangle when we multiply. For instance in carrying out \(6 \cdot 5\) we arrange eleven sticks in the following fashion

\[
\begin{array}{c}
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
6 \cdot 5 = 30
\end{array}
\]

and then count the points of intersection. This can be done by appealing either to finger-pointing or to the multiplication table. In mathematics the multiplication \(23 \cdot 5\) is the same thing as

\[(2 \cdot 10 + 3 \cdot 1) \cdot 5 = 10 \cdot 10 + 15 \cdot 1 = 1 \cdot 100 + 1 \cdot 10 + 5 \cdot 1 = 115\]

in which distributivity, associativity and commutativity are applied altogether. In the context of school we usually present the preceding calculation by the following algorithm:
What this algorithm presents is actually very simple, it presents two simple multiplications in a coherent way. The first multiplication is $3 \cdot 5 = 15$ and the second is $2 \cdot 5 = 10$. But the true meaning of the second multiplication is $20 \cdot 5 = 100$ which is marked by 10 fattened sticks. The number $15 = 10 + 5$ which is marked by 1 fattened stick and 5 normal sticks, so that altogether we have 11 fattened sticks and 5 normal sticks which is 115. The whole thing can be represented coherently, just like abacus, by the following figure:

\[
\begin{array}{c|c|c}
11 & 5 & 23 \cdot 5 = 115 \\
\hline
\end{array}
\]

The number of points of intersection in the right section of the figure is 15 which can be marked by 5 sticks and 1 fattened stick. Then we mark 5 at the bottom and move that fattened stick to the left section as showed in the figure, the number of points of intersection in the left section is 11 (fattened) which means 110. We mark 11 at the bottom. The result is thereby 115. If this way of counting holds only for $23 \cdot 5$ then it is valueless. Of course that is not the case. Let us take another example $34 \cdot 45$. As we did before, we have

\[
34 \cdot 45 = (3 \cdot 10 + 4 \cdot 1) \cdot (4 \cdot 10 + 5 \cdot 1) = 12 \cdot 100 + 15 \cdot 10 + 16 \cdot 10 + 20 \cdot 1 = 12 \cdot 100 + 31 \cdot 10 + 2 \cdot 10 + 0 \cdot 1 = 12 \cdot 100 + 3 \cdot 100 + 3 \cdot 10 + 0 \cdot 1 = 15 \cdot 100 + 3 \cdot 10 + 0 \cdot 1 = 1530
\]

which is the same thing as the following standard algorithm:

\[
\begin{array}{c|c|c}
1 & memory field for + \\
1 & memory field for \times \\
2 & memory field for \times \\
\hline
\times & 45 & multiplication field \\
\times & 34 & \\
\hline
180 & addition field \\
+ 1350 & result field \\
\hline
1530 & \\
\end{array}
\]

The algorithm puts four multiplications in one place coherently, the four multiplications are $4 \cdot 5 = 20$, $4 \cdot 4 = 16$, $3 \cdot 5 = 15$ and $3 \cdot 4 = 12$ which correspond to four corners in the following figure:
The number of points of intersection in the right section is 20, we mark 0 at
the bottom and then move two fattened sticks to the middle section. The
number of points of intersection in the middle section is 33 (meaning 330),
we mark 3 at the bottom and move 3 fattened sticks to the left section. The
number of points of intersection in the left section is 15 (meaning 1500) and
we mark 15 at the bottom. The result is 1530.

6.3 Negative Integers and Integers

Natural numbers are 1, 2, 3, · · · , and sometimes even the number 0 is called
a natural number. Natural numbers are also called positive integers, thereby
negative integers are numbers with the symbolic names:

· · · , −4, −3, −2, −1.

Positive integers, negative integers together with the number zero are called
integers. Thus integers are

· · · , −4, −3, −2, −1, 0, 1, 2, 3, 4, · · ·

Mathematics education on integers is entirely different from that on nat-
ural numbers, it is global in the sense of post-structuralism. Knowledge on
negative integers is built upon structure of natural numbers jointed by the
number zero. As a rule in mathematics the set of all natural numbers is
denoted by \( \mathbb{N} \). This set itself can not be the foundation of negative integers,
but it together with the addition and two laws (the structure of natural
numbers) offer a solid foundation. The two laws are the commutativity and
the associativity, in symbols,

- \( a + b = b + a \),
\[ a + (b + c) = (a + b) + c. \]

The ontology of a negative integer is completely different from that of a positive integer. In mathematics a positive integer is a power of a set, and powers can never lead directly to negative integers. As a result we can not treat positive integers and negative integers on an equal footing from the ontological perspective, neither in mathematics nor in mathematics education. The ontology of positive integers lies in philosophy while that of negative integers lies in mathematics itself. To be precise the ontology of negative integers lies in the structure of \( \mathbb{N} \cup \{0\} \). Need for negative integers is aroused by dissatisfaction with subtraction. A method involved in removing such dissatisfaction shows a primitive thought of relativity, and thereby is worth of a discussion here. In fact the method is well-known in advanced mathematics and I shall proceed in a standard way with a new emphasis on mathematics education.

We shall first of all treat the addition and the subtraction together from the counting perspective. The aim is to treat them symmetrically as far as possible.

When we count, we count only one number at one time as if there is no other numbers involved. A normal counting situation of counting one number, say to count the number five, can be succinctly expressed by

\[
\begin{array}{cccccc}
\text{center} & \circ & \bullet & \bullet & \bullet & \bullet & \bullet \\
0 & 1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

This diagram shows clearly, in counting the number five, that there are involved not only the number five but also the number zero, for it is after the number zero counting starts. Here I emphasized the word after because of the number zero lies outside counting though counting starts after it. The number zero is thereby the center of counting. Hence to count a number is to relate that number to (the center) the number zero in counting way, counting is consequently relative not absolute. In practice we experience the absoluteness of counting because of we automatically start our counting after the number zero. For instance when we count five, we count like this: one, two, three, four, five. In summary, counting a number is counting that number after the number zero, and it consists of

- a counting activity,
- 0 as the center of counting which is outside of counting,
- a counting relation between the number zero and that number,
- an orientation of counting.
Since the addition is a suitable combination of counting two numbers, the whole structure of natural numbers based on counting is centered at the number zero. The role of the number zero as the center of counting is caused by human habit not by counting itself. Thereby it should be removed. It is here that the subtraction enters on stage. Thus the subtraction deprives of the status of the number zero as the absolute counting center and turns each one of natural numbers into a center. It is here that a primitive thought of relativity comes in, and it is here the cunning of reason takes a turn. The syllogisms given by the cunning of reason is

\[
\text{when one counts one is counted}
\]

and

\[
\text{centralization } \rightarrow \text{de-centralization } \rightarrow \text{re-centralization}.
\]

Thus it is these syllogisms that treat the addition and the subtraction on an equal footing, which in turn also place positive integers and negative integers on a position of equal weight.

The idea behind is actually very simple. Let us take a look at the counting situation of counting the numbers 5 made above again.

In that situation, the number zero is the center of counting, for it is after there our counting starts. We start at the number one, and move to the right all the way to the number five while we count. This is the counting situation of counting the number five starting after the number zero, and is in fact the meaning of the equation

\[
0 + 1 + 1 + 1 + 1 + 1 = 5.
\]

But now let us consider another situation. Instead of starting after the number zero, we start from the number five and move to the left all the way to the number one while we count. Then we have the following counting situation of counting the number one starting from the number five

\[
5 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 1 = 0.
\]

This is a symmetric counting situation of counting the number one and the number five after/from each other.

Now we turn to a general counting situation of counting two numbers after/from each other. We take the number four and the number seven as our explanatory example.

We can express the counting situations of counting the number four and the number seven from after/each other in the diagram:
which intuitively means

\[ 4 + 1 + 1 + 1 = 7 \iff 4 + 3 = 7 \]

and

\[ 7 - 1 - 1 - 1 = 4 \iff 7 - 3 = 4. \]

Now we express the counting situations of counting the number four and the number seven after the number zero in the single graph:

These two diagrams together thereby confirm \( 4 + 3 = 7 \) and \( 7 - 3 = 4 \). Now I shall write down the new features of the counting situation of counting two numbers after/from each other. Assume \( a \) and \( b \) are two natural numbers and assume \( a < b \). Then

- in counting \( b \) after \( a \), the center is \( a \) which is outside of counting,
- the result of counting produces the positive integer \( c \) and the addition
  \[ a + c = b, \]
- in counting \( a \) from \( b \), the center is \( b \) which is inside of counting,
- the result of counting produces the negative integer \( -c \) and the subtraction
  \[ b - c = a. \]

Interchanging \( a \) and \( c \) in \( a + c = b \) we have \( c + a = b \) so that

\[ c = b - a \]

which is a usual way of defining the subtraction.

The counting way of defining subtraction and negative integers has a serious restriction, in defining \( b - a \) it is required \( a < b \). In order to remove this restriction we need to look upon addition and positive integers from a global perspective. In this perspective, the addition becomes motion in the set of all natural numbers.
As we saw above the number 3 is the counting result of counting the number 7 after the number 4. In the perspective of motion, 3 shall be considered as a motion which moves 4 to 7, and this fact is written as $3 : 4 \mapsto 7$ in which 4 is considered to be a center. Since in our treatment no center is privileged, we thereby treat all centers on an equal footing. Hence instead of considering only $3 : 4 \mapsto 7$, we also consider all possible $3 : n \mapsto n + 3$.

In this way I emphasize maximally the operational aspect of a natural number, and in doing so a natural number becomes a motion in the universe of all natural numbers. The number 3 generates a motion $n \mapsto n + 3$. A displacement of a motion at a special center $n$ is $(n + 3) - n = 3$.

In general, a motion in $\mathbb{N}$ written as $f : \mathbb{N} \to \mathbb{N}$ is a function defined on a part of $\mathbb{N}$. The addition say adding by a natural number $m$ gives rise to a natural motion $m : n \to n + m$ which is defined on the whole of $\mathbb{N}$. The wholeness is in fact a defining property of the addition by a natural number. The number 3 as motion moves 1 to 4, 2 to 5, and so on; let us write all these in a compact form

$$3 : \begin{pmatrix} n + 3 \\ \uparrow \\ n \end{pmatrix} \text{ or } \begin{pmatrix} 4 & 5 & 6 & 7 & 8 & \cdots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \end{pmatrix}.$$ 

Let us see what happens if we drop the wholeness in motion. Thus it is perfectly reasonable to consider the following motion which is defined only on a part of $\mathbb{N}$, namely, on 4, 5, 6, \ldots,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \cdots \\ 4 & 5 & 6 & 7 & 8 & \cdots \end{pmatrix}.$$ 

This motion can not be defined by a natural number through the addition, and it defines a new entity, and this new entity is $-3$. Positive integers and negative integers are consistently united in motion. The theory of negative integers based on motion is a part of mathematics and shall be presented in the rest of this section.

Insofar as there is no privileged center, displacements of a motion at all centers must be treated on an equal footing. Let us count $m_1$ at a center $m_2$ and record the result by

$$k = m_1 - m_2$$

which means that the result of counting $m_1$ at a center $m_2$ is $k$. The result of counting should be independent of center. Thereby let $n_2$ be another
center. There gives rise to a question of for which \( n_1 \) the result of counting \( n_1 \) at the center \( n_2 \) is also \( k \). That is that for which \( n_1 \) holds \( k = n_1 - n_2 \).

Therefore our starting point is two sides

\[
\begin{aligned}
  k &= m_1 - m_2 \\
  k &= n_1 - n_2
\end{aligned}
\]

where \( m_1, m_2, n_1, n_2 \) are natural numbers. The first side defines the counting situation of counting \( m_1 \) at the center \( m_2 \), while the second side defines the counting situation of counting \( n_1 \) at the center \( n_2 \). We have to investigate the way that these sides pass into each other. The two sides can be combined into a single equation \( m_1 + n_2 = n_1 + m_2 \). This third side has an important trait, the number \( k \) is suppressed explicitly, which makes the number \( k \) a construction on the whole structure of all natural numbers. Indeed the construction refers not only to the number \( k \) but can refer to other natural numbers as well, which of course paves the way for a construction of negative numbers. Let us write this syllogism as

\[
\begin{aligned}
  m_1 + n_2 &= n_1 + m_2 \\
  k &= m_1 - m_2 \\
  k &= n_1 - n_2
\end{aligned}
\]

Mathematically the third side \( m_1 + n_2 = n_1 + m_2 \) leads to a new structure on the whole collection of natural numbers. In fact the rest of this section records the becoming of this third side.

On the set \( \mathbb{N} \times \mathbb{N} \) of all ordered pairs of positive integers a relation is introduced in the following fashion. Let \((m_1, m_2)\) and \((n_1, n_2)\) be two pairs of positive integers. We say they are equivalent, in symbols,

\[
(m_1, m_2) \sim (n_1, n_2)
\]

if

\[
m_1 + n_2 = n_1 + m_2.
\]

It is a simple task to show three laws hold for such a relation. The first law is a reflexive law stating \((m_1, m_2) \sim (m_1, m_2)\), the second is a symmetric law which says that if \((m_1, m_2) \sim (n_1, n_2)\) then \((n_1, n_2) \sim (m_1, m_2)\), the third is a transitive law which states that if \((m_1, m_2) \sim (n_1, n_2)\) and if \((n_1, n_2) \sim (k_1, k_2)\) then \((m_1, m_2) \sim (k_1, k_2)\). By this equivalent relation the set \( \mathbb{N} \times \mathbb{N} \) is divided into a union of disjoint subsets called classes. Each class corresponds to a result of subtraction.

Let us take a close look at an example. Since the pair \((m_1, m_2) = (8, 6)\) is in \( \mathbb{N} \times \mathbb{N} \) it must be in some class, let us then determine this class. According
to the definition of equivalent relation mentioned above, any pair \((n_1, n_2)\) which is equivalent to \((8, 6)\) satisfies

\[8 + n_2 = n_1 + 6.\]

Numbers \(n_1, n_2\) being natural numbers, let us try first \(n_2 = 1\) then we get \(n_1 = 3\), let us try now \(n_2 = 2\) then we get \(n_1 = 4\), and so on. Continuing this way for a while we are able to draw a conclusion that all natural number solutions to the equation \(8 + n_2 = n_1 + 6\) are

\[(3, 1), \ (4, 2), \ (5, 3), \ (6, 4), \ (7, 5), \ (8, 6), \cdots .\]

Thereby, all pairs in the preceding line constitutes an equivalent class containing the pair \((8, 6)\), and the class is denoted by [\(8 \leftarrow 6\)], that is the collection of all natural number solutions to the equation \(8 + n_2 = n_1 + 6\). Thus

\[[8 \leftarrow 6] = \{(3, 1), \ (4, 2), \ (5, 3), \ (6, 4), \ (7, 5), \ (8, 6), \cdots \}.\]

Of the class let us pick a pair say \((m_1, m_2) = (3, 1)\) and we can use the same procedure to determine all pairs \((n_1, n_2)\) equivalent to \((3, 1)\), namely we can find all natural number solutions to the equation \(3 + n_2 = n_1 + 1\). Let \([3 \leftarrow 1]\) be the class of such solutions. Then

\[[3 \leftarrow 1] = \{(3, 1), \ (4, 2), \ (5, 3), \ (6, 4), \ (7, 5), \ (8, 6), \cdots \},\]

hence

\[[3 \leftarrow 1] = [8 \leftarrow 6].\]

Continuing in this way we prove that

\[[3 \leftarrow 1] = [4 \leftarrow 2] = [5 \leftarrow 3] = [6 \leftarrow 4] = [7 \leftarrow 5] = [8 \leftarrow 6] = \cdots .\]

We can now represent the result succinctly by a matrix

\[+2 : \begin{pmatrix} n + 2 \\ \uparrow \\ n \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 3 & 4 & 5 & 6 & 7 & \cdots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \end{pmatrix}.\]

Numbers in the second row are centers of counting and numbers in the first row are the results of counting. Thus, 3 is the result of adding 2 at 1, 4 is the result of adding 2 at 2, and so on. The matrix thereby defines the number 2.

If we now start with the pair \((6, 8)\) instead of \((8, 6)\), then in the exactly same way we have

\[[6 \leftarrow 8] = \{(1, 3), \ (2, 4), \ (3, 5), \ (4, 6), \ (5, 7), \ (6, 8), \cdots \},\]
\[ [1 \leftrightarrow 3] = \{(1, 3), (2, 4), (3, 5), (4, 6), (5, 7), (6, 8), \cdots \} \]

and
\[ [1 \leftrightarrow 3] = [2 \leftrightarrow 4] = [3 \leftrightarrow 5] = [4 \leftrightarrow 6] = [5 \leftrightarrow 7] = [6 \leftrightarrow 8] = \cdots \]

which gives rise to another matrix
\[
-2 : \begin{pmatrix} n \\ \uparrow \\ \uparrow \uparrow \\ \uparrow \uparrow \uparrow \\ \uparrow \uparrow \uparrow \uparrow \\ \cdots \\ \cdots \\ \cdots \\
\end{pmatrix}
\]

or
\[
\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots \\ 3 & 4 & 5 & 6 & 7 & \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\
\end{pmatrix}
\]

which defines the number \(-2\).

All these in fact can be done mathematically as follows. Now \([3 \leftrightarrow 1], [4 \leftrightarrow 2], [5 \leftrightarrow 3]\) etc are names to the same class. If there is anything that can be invariantly attached to these names then it must be the number 2, for it is this number that makes transitions \(1 \mapsto 3, 2 \mapsto 4, 3 \mapsto 5\) etc possible.

Since the meaning of the number 2 has been given it is practical to put
\[
2 = \{(3, 1), (4, 2), (5, 3), (6, 4), (7, 5), (8, 6), \cdots \}.
\]

By a similar argument it is proper to put
\[
-2 = \{(1, 3), (2, 4), (3, 5), (4, 6), (5, 7), (6, 8), \cdots \}.
\]

In general when \(n\) is a natural number we put
\[
n = \{(n + 1, 1), (n + 2, 2), (n + 3, 3), \cdots \}
\]

and
\[
-n = \{(1, n + 1), (2, n + 2), (3, n + 3), \cdots \}
\]

which clearly motivates
\[
0 = \{(1, 1), (2, 2), (3, 3), \cdots \}.
\]

We still call \(n\) positive integer, \(0\) zero, and \(-n\) negative integer. Put
\[
\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots \}
\]

and call each element in \(\mathbb{Z}\) integer.

On \(\mathbb{Z}\) three operations can be introduced. The first operation is a unary operation denoted by \(-\). Let \(k\) be an integer. If \(k = n\) is a positive integer then put
\[
-k = -n,
\]

if otherwise \(k = -n\) is a negative integer then put
\[
-k = n;
\]
and in addition put $-0 = 0$. It follows directly from the definition that

$$\neg(-k) = k$$

holds for all $k \in \mathbb{Z}$. The operation $\neg$ has a function of changing signs. It is important to notice that if $n$ is a positive integer then

$$\neg n = -n.$$

The second is a binary operation denoted by $\pm$. The operation to be defined between integers has been already defined between natural numbers. We would like to keep the similarity between the $\pm$ already defined and the $\pm$ to be defined as much as we could. Thus since

$$2 = \{(2 + 1, 1), (2 + 2, 2), (2 + 3, 3), (2 + 4, 4), (2 + 5, 5), (2 + 6, 6), \cdots \},$$
$$5 = \{(5 + 1, 1), (5 + 2, 2), (5 + 3, 3), (5 + 4, 4), (5 + 5, 5), (5 + 6, 6), \cdots \},$$
$$7 = \{(7 + 1, 1), (7 + 2, 2), (7 + 3, 3), (7 + 4, 4), (7 + 5, 5), (7 + 6, 6), \cdots \},$$
we can define $\pm$ in $2 + 5$ by $\pm$ in $2 + 5 = 7$ in the following way. We begin with assuming that two integers we would like add them have the same sign. Thus assume for example

$$k_1 = \{(k_1 + 1, 1), (k_1 + 2, 2), (k_1 + 3, 3), \cdots \},$$
$$k_2 = \{(k_2 + 1, 1), (k_2 + 2, 2), (k_2 + 3, 3), \cdots \}$$

are two positive integers. Then put

$$k_1 + k_2 = \{(k_1 + k_2 + 1, 1), (k_1 + k_2 + 2, 2), (k_1 + k_2 + 3, 3), \cdots \}.$$  

Here are an example. Since

$$5 = \{(6, 1), (7, 2), (8, 3), \cdots \},$$
$$7 = \{(8, 1), (9, 2), (10, 3), \cdots \},$$
$$-5 = \{(1, 6), (2, 7), (3, 8), \cdots \},$$
$$-7 = \{(1, 8), (2, 9), (3, 10), \cdots \},$$
then

$$5 + 7 = \{(13, 1), (14, 2), (15, 3), \cdots \} = 12$$

and

$$-5 + (-7) = \{(1, 13), (2, 14), (3, 15), \cdots \} = -12.$$  

Assume now that two integers we would like add them have different signs. Thus assume that

$$k_1 = \{(k_1 + 1, 1), (k_1 + 2, 2), (k_1 + 3, 3), \cdots \},$$
\[-k_2 = \{(1, k_2 + 1), (2, k_2 + 2), (3, k_2 + 3), \cdots \}\]

are two integers to be added. If \(k_1 \geq k_2\) then put

\[k_1 + (-k_2) = \{(k_1 - k_2 + 1, 1), (k_1 - k_2 + 2, 2), (k_1 - k_2 + 3, 3), \cdots \},\]

or else if \(k_1 < k_2\) then put

\[k_1 + (-k_2) = \{(1, k_2 - k_1 + 1), (2, k_2 - k_1 + 2), (3, k_2 - k_1 + 3), \cdots \}.\]

We now complete our definition of the addition by postulating \(k + 0 = 0 + k = k\). It follows from the definition that \(k + (-k) = 0\) and

\[5 + (-7) = \{(1, 3), (2, 4), (3, 5), \cdots \} = -2.\]

The subtraction can be constructed of the two operations defined above. Thus let \(k_1\) and \(k_2\) be two integers. We define

\[k_1 - k_2 = k_1 + (-k_2).\]

Since

\[7 = \{(8, 1), (9, 2), (10, 3), \cdots \},\]

then

\[\neg 7 = \{(1, 8), (2, 9), (3, 10), \cdots \} = -7,\]

so that

\[5 - 7 = 5 + (-7) = 5 + (-7) = -2.\]

The third operation is also a binary operation denoted by \(\cdot\). Let

\[k_1 = \{(k_1 + 1, 1), (k_1 + 2, 2), (k_1 + 3, 3), \cdots \},\]

\[k_2 = \{(k_2 + 1, 1), (k_2 + 2, 2), (k_2 + 3, 3), \cdots \}\]

be two positive integers. Then put

\[k_1 \cdot k_2 = \{(k_1 \cdot k_2 + 1, 1), (k_1 \cdot k_2 + 2, 2), (k_1 \cdot k_2 + 3, 3), \cdots \}.\]

If \(k_1\) and \(k_2\) have negative sign then put

\[k_1 \cdot k_2 = (-k_1) \cdot (-k_2).\]

If \(k_1\) and \(k_2\) have different signs, say \(k_1\) is positive while \(k_2\) is negative, then put

\[k_1 \cdot k_2 = -(k_1 \cdot (-k_2)).\]

At last we complete the definition of multiplication by postulating \(k \cdot 0 = 0 \cdot k = 0\).
It is tiresome to verify associativity, commutativity and distributivity for the addition and the multiplication defined on \( \mathbb{Z} \). Mathematicians summarize all of these by a single sentence that \( \mathbb{Z} \) is a ring. When the construction of integers has been carried out, we can change \( k \) back to \( k \).

Now that we know the construction of integers, we can make some remarks on them. The structure of integers is built upon the structure of natural numbers, in a sense, it is a post-structure of natural numbers. The construction is not a local one, in fact it depends on the whole structure of natural numbers. Knowledge of the number 5 does not lead to that of \(-5\). Inasmuch as the construction of integers is utterly different from that of natural numbers, mathematics education on the number \(-5\) differs a great deal from that on the number 5. A popular method of teaching by making use of metaphors is very important at the beginning of encounter with negative integers. The limitations of the method must be aware of constantly. A mathematics education design on integers is not toward ontology but toward ontology in natural numbers, and this makes knowledge on integers more mathematical than philosophical. A teaching practitioner on integers must be aware of a danger of falling into a vicious cycle and must attentively distinguish premises from conclusions. A typical case is a popular saying that minus times minus becomes plus. I can think of two ways to interpret this saying. The first is \((-) \cdot (-) = +\) which does not make any mathematical sense. The closest we can get is

\[ -(\neg k) = k. \]

As far as I can see, this is a definition. If an even harder attempt is made to extract a mathematical thing then it must be \((-1) \cdot (-1) = +1\). But this is precisely the second interpretation I can think of, namely, \((-3) \cdot (-4) = 3 \cdot 4\) which is a special case of a general formula

\[ (-m) \cdot (-n) = m \cdot n. \]

This is again a definition for \(-m = -m\) and \(-n = -n\). Definitions are not science but still can be good. The definition we adopted newly is good for it is in conformity with the harmony in the universe \( \mathbb{Z} \). The harmony is expressed in terms of the three laws and \( 0 \cdot k = k \cdot 0 = 0 \).

I shall now show that \((-m) \cdot (-n) = m \cdot n\) is a consequence of the harmony. Since

\[ 0 = (-m) + m, \]

then

\[ 0 = 0 \cdot n = ((-m) + m) \cdot n = (-m) \cdot n + m \cdot n \]

which gives

\[ 0 + (-(m \cdot n)) = ((-m) \cdot n + m \cdot n) + (-(m \cdot n)) \]
which is
\[ -(m \cdot n) = (-m) \cdot n + (m \cdot n + (- (m \cdot n))) = (-m) \cdot n \]
which results in
\[ (-m) \cdot n = -(m \cdot n). \]
Since again
\[ 0 = (-m) + m, \]
then
\[ 0 = ((-m) + m) \cdot (-n) = (-m) \cdot (-n) + m \cdot (-n). \]
It follows from what just proved that
\[ 0 = (-m) \cdot (-n) + (- (m \cdot n)) \]
which gives
\[ 0 + m \cdot n = (-m) \cdot (-n) + (- (m \cdot n)) + m \cdot n \]
so that
\[ (-m) \cdot (-n) = m \cdot n. \]
Therefore in defining \((-m) \cdot (-n)\) either we define it as \(m \cdot n\) or else define it as any number in conformity to the harmony of the entire ring \(Z\), both ways lead to the same number.

### 6.4 Fractions

Mathematics education on fractions is a challenge. Intuition needed to intuit fractions is much poorly developed in comparison to the formalization of fractions. Educational designs on fractions should begin with a revaluation of intuition appearing in acquisition of the concept of natural numbers, for that intuition has limited power in the domain of fractions. Any intuition suitable for fractions should be oriented towards the formalization made in the text below. Fractions are a global construction on the entire ring of integers.

In the ring of integers counting can be centered at any integer, and a center is an additive center. With respect to the multiplication the number 1 is the privileged unit (the multiplicative center) to the effect that any number can be measured by it. If \(n\) is a number then the set of all \(k \cdot n\) where \(k\) runs through all integers is called a gauge (scale) determined by \(n\). Thus the gauge containing 1 is the entire \(Z\). Parallel to that the deprivation of 0 as the only additive center gives rise to the subtraction and negative integers, the deprivation of 1 as the only multiplicative center gives rise to the division and fractions. In the field of fractions any non-zero number can
be a unit so as to a number can be measured by a non-zero number. So realized, a transition to fractions from integers is entirely parallel to that from natural numbers to integers. Let $\mathbb{Z}^*$ be the set of all possible units, namely, it consists of all non-zero integers. We shall measure each integer in $\mathbb{Z}$ by each integer in $\mathbb{Z}^*$. Thereby the syllogism

when one measures one is measured

replaces the syllogism

when one counts one is counted

which is a principal syllogism in dealing with natural numbers.

A normal measuring situation of measuring a number, say to measuring 5 by 1, can be expressed by

\[
\begin{array}{c}
\text{5} \\
\cdot \\
\text{1} \\
\end{array}
\]

which (the number 5 in the left side is the result of measuring) is

\[5 \cdot 1 = 5.\]

In this measuring, the shorter one (the ruler) divides the longer one. The same as the last section, measuring a number consists of

- a measuring activity,
- a measuring relation between the number one and that number,
- an orientation of the measuring.

Now let us change the order of measuring, that is let us look at a new measuring situation:

\[
\begin{array}{c}
\text{1} \\
\cdot \\
\text{5} \\
\end{array}
\]

The result of measuring 1 by 5 is $1/5$ and

\[\frac{1}{5} \cdot 5 = 1.\]

And in this measuring, the longer one (the ruler) divides the shorter one which amounts to the same thing as the ruler is divided by that to be measured by the ruler itself.

Let $m$ be an integer and $n$ be a non-zero integer (unit). In measuring $m$ by $n$ we search mathematically for a definite number $r$ so as to

\[m = r \cdot n.\]
The equation means that the result of measuring \( m \) by \( n \) is \( r \). To measure is to be measured, and to measure is to compare gauges. Therefore the result \( r \) of measuring depends only upon the gauges defined by \( m \) and \( n \), for the equation \( m = r \cdot n \) gives rise to

\[
(km) = r \cdot (kn).
\]

For instance 2 is the result of measuring 4 by 2 or 6 by 3 or in general the gauge \( 2k \) by the gauge \( k \), namely,

\[
(2k) = 2 \cdot (k).
\]

It is consequently desirable to determine the dependence of the number \( r \) upon \( m, n \) when \( n \) varies. Hence we are not only interested in one equation of \( m = r \cdot n \) but two such equations (two sides):

\[
\begin{align*}
\{ m_1 &= r \cdot n_1 \\
\quad m_2 &= r \cdot n_2.
\end{align*}
\]

The first side defines a measuring situation of measuring \( m_1 \) by \( n_1 \), while the second side defines another measuring situation of measuring \( m_2 \) by \( n_2 \). A synthesis of them gives rise to an equation \( m_1 \cdot n_2 = m_2 \cdot n_1 \) which is the third side. Let us write this syllogism as

\[
\begin{tikzpicture}
  \node (m1) at (0,0) {\( m_1 \cdot n_2 = m_2 \cdot n_1 \)};
  \node (m1n1) at (2,-1) {\( m_1 = r \cdot n_1 \)};
  \node (m2n2) at (-2,-1) {\( m_2 = r \cdot n_2 \)};
  \draw (m1n1) -- (m1) -- (m2n2);
\end{tikzpicture}
\]

Mathematically the third side defines a new structure. On the set \( \mathbb{Z} \times \mathbb{Z}^* \) of all ordered pairs of integers \( (m, n) \) with \( n \neq 0 \), a relation is introduced in the following fashion. Let \( (m_1, n_1) \) and \( (m_2, n_2) \) be such two pairs. We say that they are equivalent, in symbols,

\[
(m_1, n_1) \sim (m_2, n_2)
\]

if

\[
m_1 \cdot n_2 = m_2 \cdot n_1.
\]

It is a simple task to show that three laws hold for such a relation. The first law is a reflexive law stating \( (m, n) \sim (m, n) \), the second is a symmetric law which says that if \( (m_1, n_1) \sim (m_2, n_2) \) then \( (m_2, n_2) \sim (m_1, n_1) \), the third is a transitive law which states that if \( (m_1, n_1) \sim (m_2, n_2) \) and if \( (m_2, n_2) \sim (m_3, n_3) \) then \( (m_1, n_1) \sim (m_3, n_3) \).

By this equivalent relation the set \( \mathbb{Z} \times \mathbb{Z}^* \) is divided into a union of disjoint subsets called classes. Each class corresponds to a result of division.
Let us take a close look at an example. Since the pair \((m_1, n_1) = (8, 6)\) is in \(\mathbb{Z} \times \mathbb{Z}^*\) it must be in some class, let us then determine this class. According to the definition of equivalent relation mentioned above, any pair \((m_2, n_2)\) equivalent to \((8, 6)\) satisfies
\[
8n_2 = 6m_2.
\]

Numbers \(m_2, n_2\) being integers with \(n_2 \neq 0\), let us try first \(n_2 = 1, 2\) then we have no solution, if we try \(n_2 = 3\) then \(m_2 = 4\), let us try now \(n_2 = 4, 5\) then we have no solution, if we try \(n_2 = 6\) then \(m_2 = 8\), and so on. Continuing this way for a while we are able to draw a conclusion that all integer solutions to the equation \(8n_2 = 6m_2\) are
\[
\cdots, (-12, -9), (-8, -6), (-4, -3), (4, 3), (8, 6), (12, 9), \cdots.
\]

We can express all these solutions by a single expression \((4k, 3k)\) where \(k\) run through all non-zero integers. Now we define
\[
\frac{8}{6} = \{\cdots, (-12, -9), (-8, -6), (-4, -3), (4, 3), (8, 6), (12, 9), \cdots \}.
\]

If on the other hand we pick another pair \((4k, 3k)\) from the class where \(k\) is some integer and let \((m_1, n_1) = (4k, 3k)\), and if we run what we just did once more, then we conclude that
\[
\frac{4k}{3k} = \{\cdots, (-12, -9), (-8, -6), (-4, -3), (4, 3), (8, 6), (12, 9), \cdots \}.
\]

In general, let \(m\) and \(n\) be integers with \(n \neq 0\) then we define \(m/n\) to be the equivalent class of \(\mathbb{Z} \times \mathbb{Z}^*\) which contains the pair \((m, n)\).

The example showed above proves
\[
\cdots = \frac{-12}{-9} = \frac{-8}{-6} = \frac{-4}{-3} = \frac{4}{3} = \frac{8}{6} = \frac{12}{9} = \cdots,
\]
for each of them refers to the same class. We represent the findings by a matrix
\[
(\times) \begin{pmatrix} 8 \\ 6 \end{pmatrix} : \begin{pmatrix} 4n \\ 3n \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cdots & -12 & -8 & -4 & 4 & 8 & 12 & \cdots \\ \cdots & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \cdots \end{pmatrix}.
\]

Thus the fraction 8/6 gives rise to a motion in the set \(\mathbb{Z} \setminus \{0\}\).

Let us look at the definition of fractions closely. Thus let \((m, n)\) and \((m_1, n_1)\) be in the same class. Being in the same class it follows that
\[
\frac{m}{n} = \frac{m_1}{n_1},
\]
for both fractions refer to the same class. And also being in the same class it holds $m \cdot n_1 = m_1 \cdot n$ which is the defining condition for the very same class. We can write them together shortly and concisely

$$\frac{m}{n} = \frac{m_1}{n_1} \iff m \cdot n_1 = m_1 \cdot n,$$

for each time when we look at the two equations we are being referred to that same class of which the two equations are two sides.

When the ruler of measuring is the number one, then in conformity with counting I shall write

$$\frac{m}{1} = m.$$

Now that we have defined fractions, we shall discuss operations we can supply fractions with. First of all, since $m \cdot (cn) = (cm) \cdot n$ then

$$\frac{m}{n} = \frac{cm}{cn}$$

holds for all non-zero integers $c$ and $n$.

A fraction is the result of measuring $m$ by $n$, mathematically this means

$$m = \left(\frac{m}{n}\right) \cdot n.$$

But a definition of multiplication between two fractions has not yet been given. This is our next task. The multiplication is already defined on integers which, after identification $m = m/1$, is also the multiplication on fractions of this sort. Thus

$$\left(\frac{m}{1}\right) \cdot \left(\frac{n}{1}\right) = \frac{m \cdot n}{1}.$$

We want the multiplication to be defined on fractions is the same as that when fractions are just integers. In fact I shall go the other way around, I shall try to get some ideas by examining the multiplication on integers from the perspective of fractions. To this aim let $m_1$ and $m_2$ be integers. By the identification $m_1 = m_1/1$ and $m_2 = m_2/1$ integers become fractions. Since

$$\frac{m_1}{1} = \frac{m_3}{n_3} \iff m_1 \cdot n_3 = m_3 \cdot 1,$$

and

$$\frac{m_2}{1} = \frac{m_4}{n_4} \iff m_2 \cdot n_4 = m_4 \cdot 1,$$

then

$$\frac{m_1 \cdot m_2}{1} = \frac{m_3 \cdot m_4}{n_3 \cdot n_4} \iff m_1 \cdot m_2 \cdot n_3 \cdot n_4 = m_3 \cdot m_4 \cdot 1.$$

Hence if we insist

$$\left(\frac{m_1}{1}\right) \cdot \left(\frac{m_2}{1}\right) = m_1 \cdot m_2 = \frac{m_1 \cdot m_2}{1};$$
then we must have
\[
\left(\frac{m_3}{n_3}\right) \cdot \left(\frac{m_4}{n_4}\right) = \frac{m_3 \cdot m_4}{n_3 \cdot n_4}.
\]
Hence it is natural to define a multiplication of two fractions by
\[
\left(\frac{a}{b}\right) \cdot \left(\frac{c}{d}\right) = \frac{a \cdot c}{b \cdot d}.
\]

With this definition of multiplication we have
\[
\left(\frac{m}{n}\right) \cdot \left(\frac{n}{1}\right) = \frac{n \cdot m}{n \cdot 1} = \frac{m}{1},
\]
so that
\[
m = \left(\frac{m}{n}\right) \cdot n.
\]

What we have showed so far is a fact to the effect that a fraction \(m/n\) is a solution to the equation
\[
x \cdot n = m.
\]

We shall now show that any fraction solution to the same equation is equal to \(m/n\). Indeed let \(p/q\) be another solution. Then
\[
\left(\frac{n}{1}\right) \cdot \left(\frac{p}{q}\right) = \frac{m}{1}
\]
which is
\[
\frac{np}{q} = \frac{m}{1} \iff np = mq \iff \frac{p}{q} = \frac{m}{n}.
\]

Let us summarize what we have done in a theorem.

**Theorem.** Let \(m\) and \(n\) be two integers with \(n \neq 0\). Then the equation \(nx = m\) has always a solution in fractions and in addition the solution is unique.

We shall now use this theorem to give a reasonable definition of addition and division of fractions. Thus let \(a/b\) and \(c/d\) be two fractions. According to the theorem
\[
b \cdot \left(\frac{a}{b}\right) = a, \quad d \cdot \left(\frac{c}{d}\right) = c.
\]

Multiplying the first equation by \(d\), the second by \(b\) and then adding or subtracting them, we get
\[
(bd) \cdot \left(\frac{a}{b} \pm \frac{c}{d}\right) = ad \pm cb.
\]
Since this equation has another solution \((ad \pm cb)/(bd)\) and since the solution to the equation is unique, then
\[
\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}.
\]
Now we turn our attention to the division. There are at least two ways to motivate such a definition. The first is to make use of $m/n = cm/cn$ and the second is to use the theorem. We have first

$$\frac{a/b}{c/d} = \frac{a/b \cdot d/c}{c/d} = \frac{ad/bc}{cd/cd} = \frac{ad}{bc}. $$

The fraction $(a/b)/(c/d)$ is a solution to the equation

$$ \left(\frac{c}{d}\right) \cdot x = \frac{a}{b} $$

which has another solution $ad/bc$ for

$$ \frac{c}{d} \cdot \frac{ad}{bc} = \frac{cd \cdot a}{cd \cdot b} = \frac{a}{b}, $$

it follows from the uniqueness of the solution that

$$ \left(\frac{a}{b}\right)/\left(\frac{c}{d}\right) = \frac{ad}{bc}. $$

The three laws hold as well for the operations on fractions, they follow from the corresponding laws within integers. All these facts are summarized by saying that the set $\mathbb{Q}$ of all fractions is a field.

Mathematics education on fractions is a great challenge. It would be pretentious to think of the problem can be solved easily. Although a huge amount of didactical works have been done in this area, there is still a need to deepen the knowledge and to widen the horizon. Teaching practitioners must be constantly aware of running into the case of a faulty formula

$$ \frac{2}{3} + \frac{3}{4} = \frac{5}{7}. $$

The one who writes such a thing definitely knows that he is doing a hopeless guessing, and he has very thin knowledge about the three fractions there. Knowledge on a fraction say $2/3$ can start from different perspectives but they all end at that mathematical determination

$$ 3 \cdot \left(\frac{2}{3}\right) = 2. $$

It is this determination together with the three laws that makes the addition safe. In fact, since the preceding equation determines $2/3$, and since

$$ 4 \cdot \left(\frac{3}{4}\right) = 3; $$

then we have

$$ 4 \cdot 3 \cdot \left(\frac{2}{3}\right) = 4 \cdot 2, \quad 3 \cdot 4 \cdot \left(\frac{3}{4}\right) = 3 \cdot 3. $$
Adding these two equalities and making use of the distributivity, we get

\[ 12 \cdot \left( \frac{2}{3} + \frac{3}{4} \right) = 8 + 9 = 17, \]

so that

\[ \frac{2}{3} + \frac{3}{4} = \frac{17}{12}. \]

Educational difficulties on the fractions are caused by a continuation of customary use of intuition. By the time when fractions enter on the educational stage, intuition has given us a solid knowledge on counting as well as integers. Intuition to be used to intuit integers is very different from that to be used to intuit fractions. If one uses the same way of intuiting integers and fractions then it is a matter of time when confusions show up. To my knowledge there are some special ways of intuiting fractions but not a united way. A shift among special ways often causes even greater confusions, for the ways are often personal. Formalization of fractions on the other side is a clear-cut case, simple to use once it is mastered, and should be an aim of education.

Like the concept of natural numbers, the concept of fractions also lies in the unity of necessity and contingency. This unity is not the only one that fractions lie in, there is another unity in which fractions lie. It is the unity of whole and part. It has been known for a long time that the unity of part and whole causes a great deal of learning difficulties. I used to pose a question, “There are two fathers and two sons, they eat three apples and each one eats precisely one apple, how do they manage to do that?” to my students. In fact many do not know the answer. Of course the story of three brothers which was used by Piaget and his group belongs to the same category. The point is that a part lies in the whole, so that if I count things in the part and count things in the whole then I have counted things in part twice. That causes a confusion. A common mistake is to count things not in the part when count things in the whole. Persons who count like this argue that we have just counted things in the part. This problem is more psychological than mathematical to which I offer no comment, though I know that I should be very careful when I teach this kind of subject.

The mathematical structure of fractions presented above is a final aim of mathematics education on fractions, but it is not the subject where teaching and learning begins with. There is no unique way of starting with fractions, in fact, several options are available. In the rest of present section I shall discuss some of these options.

The concept of fraction is a synthesis of part and whole, and it describes a proportional relationship between part and whole. Thus 1/3 describes such a relationship between part and whole that three parts together is the whole, or what amounts to the same thing, that if you divide the whole into three equal parts then each part is 1/3. The following figure shows certain
aspect about 1/3

If we take a part as \( \bullet \) and take the whole as \( \circ \cdot \circ \), then three parts give the whole and then 1/3 gives a relationship between the black ball and the whole \( \circ \cdot \circ \). It takes some time to really grasp the idea which the picture contains. However, once one understands this, some interesting phenomena show up. The picture

\[
\circ \circ \circ \bullet \bullet \circ \circ\circ
\]

shows the same relationship between the black balls and the whole \( \circ \circ \circ \bullet \bullet \circ \circ \circ \), and the picture

\[
\circ \circ \circ \bullet \bullet \bullet \circ \circ \circ \circ \circ \]

also shows the same relationship between the black balls and the whole \( \circ \circ \circ \bullet \bullet \bullet \circ \circ \circ \circ \circ \), and so on. What the pictures have showed us is simply relations among seemingly different fractions

\[
\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \cdots,
\]

so that a choice of 1/3 is contingent. This is not an incident and any choice of a fraction is contingent because of

\[
\frac{m}{n} = \frac{2m}{2n} = \frac{3m}{3n} = \cdots.
\]

Therefore we must take into consideration of the law of contingency and necessity side by side with the law of part and whole.

The faulty formula mentioned above is possible because of lacking efficient knowledge on these two laws. Above I offered a formalized way of avoiding the fault, and now I shall supply with an intuitive way of doing that, and the intuitive way is in one way or other from the counting perspective.

The definition of multiplication of two natural numbers is

\[
m \cdot n = \underbrace{n + n + \cdots + n}_{m}.
\]

Since

\[
m \cdot \left( \frac{1}{n} \right) = \left( \frac{m}{1} \right) \cdot \left( \frac{1}{n} \right) = \frac{m}{n},
\]

it is reasonable to interpret \( m/n \) as

\[
\frac{m}{n} = \underbrace{\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{m}
\]

which means that we treat \( m/n \) as the counting result of counting the number of 1/n treated as a unit. For instance,

\[
\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}.
\]
I am now ready to discuss the above-mentioned fault once again. The fraction $2/3$ means two of $1/3$ and $3/4$ means three of $1/4$. The fractions $1/3$ and $1/4$ as units are not commensurable, thereby they can not be added together. The situation is entirely similar to that of apple and book as units. In order to proceed we must invoke the law of contingency and necessity. As long as fractions concern,

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12}$$

and

$$\frac{3}{4} = \frac{6}{8} = \frac{9}{12}.$$ 

The formulas show that $2/3$ also means eight of $1/12$ and $3/4$ means nine of $1/12$. Now $1/12$ is the only unit, it follows that

$$\frac{2}{3} + \frac{3}{4} = \frac{8}{12} + \frac{9}{12} = 8 \cdot \left(\frac{1}{12}\right) + 9 \cdot \left(\frac{1}{12}\right) = 17 \cdot \left(\frac{1}{12}\right) = \frac{17}{12}.$$

After interpreting $m/n$ as the result of counting the unit $1/n$, it is rather easy to give a rule on multiplication of a fraction $m/n$ by an integer $k$. In fact,

$$k \cdot \frac{m}{n} = k \cdot \frac{m}{n} + \cdots + k \cdot \frac{m}{n} = k \cdot \left(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\right) = km \cdot \frac{1}{n} = \frac{km}{n},$$

that is

$$k \cdot \frac{m}{n} = \frac{km}{n}.$$ 

When we added $3/2 + 3/4$, we had two units $1/2$ and $1/4$. These two units are connected by $1/2 = 2 \cdot (1/4)$ in which case these two units are said to be commensurable. In adding two fractions with incommensurable units, we need a third unit which is commensurable with the first two units. Thus in adding $a/b + c/d$, we have firstly two units $1/b$ and $1/d$. Now we create the third unit $1/bd$. This is a simple creation under the requirement that the third one is commensurable to both $1/b$ and $1/d$. Since

$$\frac{1}{b} = d \cdot \frac{1}{bd'}, \quad \frac{1}{d} = b \cdot \frac{1}{bd'},$$

the requirement is fulfilled. Now

$$\frac{a}{b} + \frac{c}{d} = a \cdot \frac{1}{b} + c \cdot \frac{1}{d} = ad \cdot \frac{1}{bd} + bc \cdot \frac{1}{bd} = (ad + bc) \cdot \frac{1}{bd} = \frac{ad + bc}{bd},$$

that is

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$
I shall now end the present section by giving yet another perspective of adding two fractions. I shall use $\frac{1}{3} + \frac{2}{5}$ as my explanatory example.

As we saw above, the fraction $\frac{1}{3}$, from the counting perspective, means one of $\frac{1}{3}$ treated as a unit; while $\frac{2}{5}$ means two of $\frac{1}{5}$ treated as another unit. These two units can not be compared at the moment, thereby the third unit is needed. The new perspective is to replace unit by dimension.

The fraction $\frac{2}{5}$ can be represented by a matrix made of black balls and white balls $\bullet \circ \circ \circ \circ$. We say that the dimension of it is $1 \times 5$. In general, for a matrix of black balls and white balls, we call $m \times n$ dimension of it, if $m$ is the number of rows and $n$ is the number of columns. With respect to dimensions, we shall not make distinction between $m \times n$ and $n \times m$, that is we adopt

$$m \times n = n \times m.$$ 

Thus the fraction $\frac{2}{5}$ can be represented as matrices of different dimensions, and it is a variant of the principle of cloning measuring. For instance

\[
\begin{array}{cccc}
5 \times 1 & 5 \times 2 & 5 \times 3 & 5 \times 4 \\
\bullet & \bullet & \bullet & \bullet \\
\frac{2}{5} & \frac{4}{10} & \frac{6}{15} & \frac{8}{20} \\
\end{array}
\]

Likewise the fraction $\frac{1}{3}$ can also be represented as

\[
\begin{array}{cccc}
3 \times 1 & 3 \times 2 & 3 \times 3 & 3 \times 4 & 3 \times 5 \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\frac{1}{3} & \frac{2}{6} & \frac{3}{9} & \frac{4}{12} & \frac{5}{15} \\
\end{array}
\]

Now we lay out some rules for the addition of two fractions. They are

1. two fractions can be added only if they have the same dimension,
2. the addition of two fractions of the same dimension consists in keeping the dimension and adding the black balls appearing in both fractions.

Thus in adding $\frac{1}{3} + \frac{2}{5}$, we have

\[
\frac{1}{3} + \frac{2}{5} = \quad \frac{11}{15}
\]
6.5 Numbers in Triangular Structure

Teaching and learning must be united in one and the same ever going process. The process is a synthesis of normal period against singular period and of structure against de-structure. A normal period is long, painless and filled with tasks of problem-solving which end with more or less predictable results; whereas a singular period is short, painful and filled with challenges, unpredictable connections and deep satisfactions. Replacing first principles, breaking with habits, constructing new structures, confronting fear, frustrating, confusing are typical in periods of transition. In a process things are organized and re-organized, invented and re-invented with orientation. Once a direction is determined, somethings fall into the focus of our attention while others are side-pushed. Things in the focus are determined not only by the student’s ability to invent and desire for progress but also by the teacher’s assistance to re-invent and experience. In addition students are supplied with opportunity of lifting themselves in such a way that their horizon and their eyes are opened wider and wider and still wider. The syllogism directs the development of this section is

\[
\text{structure} \rightarrow \text{de-structure} \rightarrow \text{re-structure}.
\]

The path along which we proceeded is natural numbers, integers and fractions. Each class constitutes its own structure, and all these structures are global. Once a global structure has been constructed, all kinds of local structure enter on the developmental stage. I shall discuss one kind of local structure in this section.

Already in the antiquity mathematicians were interested in numbers of points in triangles in the figures:

\[
\begin{array}{ccccccccccc}
\cdot & . & . & . & . & . & . & . & . & . & . & \ldots \\
\end{array}
\]

Mathematically they were interested in numbers having structure

\[
1, \quad 1 + 2, \quad 1 + 2 + 3, \quad 1 + 2 + 3 + 4, \quad \cdots.
\]

A most general sum of this type is

\[
1 + 2 + 3 + \cdots + n
\]

where \( n \) is some natural number. A sum \( 1 + 2 + 3 + \cdots + n \) is a local structure. Numbers 1, 3, 6, 10, 15 have this structure but 2, 4, 5 do not have. Mathematicians of the antiquity found an interesting expression for numbers of this structure, it is

\[
1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}.
\]
With this formula we can very quickly judge whether a given number has that structure. Putting \( n = 35 \) we judge that the number \( 35 \cdot 18 = 630 \) has the structure. On the other hand, in judging whether the number 830 has the structure, we need to solve the equation

\[
\frac{n(n + 1)}{2} = 830.
\]

If the equation has a natural number solution then the number 830 has the structure otherwise has not. The preceding equation can be written as a quadratic equation

\[
n^2 + n - 1660 = 0
\]

whose solutions are

\[
n = \frac{-1 \pm \sqrt{(\frac{1}{2})^2 + 1660}}{2} = \frac{-1 \pm \sqrt{6641}}{2}
\]

which are not natural numbers. Then the number 830 has not the structure.

A special case of the above-mentioned formula became popular because of Gauss. It is said that soon after the teacher gave Gauss a task of computing \( 1 + 2 + 3 + \cdots + 100 \), he supplied with the answer 5010 and explained the way he did. He did not add the sum in the usual way, that is

\[
1 + 2 + 3 + 4 + 5 + \cdots + 100 = 3 + 3 + 4 + 5 + \cdots + 100 = 6 + 4 + 5 + \cdots + 100
\]

and so on, instead, he did by adding the first number with the last number that is \( 1 + 100 \) and by adding the second number with the next to the last number that is \( 2 + 99 \) etc. He argued that the results of this kind of sum are always 101 and there are exactly 50 of those, so that \( 50 \cdot 101 = 5050 \) is the result. Many know of this piece of history. I saw once a dramatized version on a program about teaching on Swedish national channel SVT 1. I have many reasons to believe that this story has been told many times by many teachers to their students. I also believe this is a good example about teaching and learning which is worth of some discussion.

When can a teacher take the example in his class? This is easy to answer. As long as students can make some simple addition and time is permitted. If time is not enough one can add \( 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \). The simplified version does not spoil the problem. The degree of difficulty is the same. However the teacher can not expect the students to add like Gauss did, the students add usually in normal way. In fact I did my own empirical study on the matter.

I asked, at several occasions, some children of nine years old to add 1 with 2 then with 3 then with 4 and so on all the way up to 10. All of them added according to the order \( 1 + 2 = 3, \ 3 + 3 = 6, \ 6 + 4 = 10 \) etc and got the right result that is 55.
At some other occasions, I asked some other children of the same age to add 10 with 9 then with 8 then with 7 and so on all the way down to 1. I was surprised to see that the children of the second group added in the same fashion as the children of the first group, that is $1 + 2 = 3$, $3 + 3 = 6$, $6 + 4 = 10$ etc.

Why did not they add like $10 + 9 = 19$, $19 + 8 = 27$, $27 + 7 = 34$ etc? There must be several answers to this question. One of the possible answers is that children did so for they were used to. They had established such a behavior already. Education can lift children from a level to another, and within each level behavior is ruled over by the law of inertia. At the beginning the child confronts an object which is not cognized both perceptually and structurally, through education not only did the child understand perceptually the object but also the object is classified according to its structure by the child. Once a structure is established, it becomes a harbor where the child anchors his mind with both psychological and philosophical safety. Problem-solving by making use of the structure becomes sightseeing on the landscape of the harbor. This is a happy moment for the child and the moment is ruled by the law of inertia, the moment is good if it is short and bad if it is long.

Now back to that sum $1 + 2 + \cdots + 10$. In order to see how far I could go, I chose a girl from the first group. I said to her: “what you did before was correct and there is however another way to do it, would you like we together make a try?”. She wanted to try. Then I said to her: “please write down $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ in a row on a sheet of paper”. She did that. Then I said to her: “please point at the number one with one of fingers of your left hand and at the same time point at the number ten with one of fingers of your right hand and then add them”. She added and said immediately: “eleven”. I said to her: “that is very good and now move that finger which points for the moment at the number one one step to the right and at the same time move that finger which points for the moment at the number ten one step to the left and then add them”. She added and said immediately: “eleven”. I said to her: “that is very good and now move that finger which points for the moment at the number one one step to the right and at the same time move that finger which points for the moment at the number ten one step to the left and then add them”. She added and said with the same speed as before: “eleven”. I continued and said to her: “you can probably guess what I want you to do next, right?”. She said: “the same kind of thing?” I said to her: “yes”. She did that and said: “eleven”, “eleven”, “eleven”. At the end she could not move her fingers further any more, and withdrew her hands away from the sheet of paper.

I began to notice that she threw her glance alternatively at me and at those ten numbers on that sheet of paper. I could not help to notice a mixed expression of light and shadow on her face, and asked her: “what are you thinking now?”. “I do not know and this eleven, hm”, she replied. I waited for a while and said to her again: “do you remember the reason that we started the whole thing?”. She said: “you wanted me to add one with two and then with three and now I know (before she finished her sentence) I have five elevens that make fifty-five”. “Very good and tell me what you have done”, I said to her. She said: “I added one and ten first and got eleven, then
I added two and nine and got eleven, and so on, finally I got five elevens and five times eleven is fifty-five”. I was glad at the progress she made and wanted to help her to see the whole thing from a non-mathematical perspective, and I said to her: “you see, last time when you added these ten numbers you added them by a pattern that you were used to, and this time, when you added these ten numbers you added them by a new pattern, both patterns are useful, but sometimes one pattern is better than other pattern, in adding $1+2+\cdots+10$, the second pattern is better than the first, numbers are just like shapes of mountain or forms of lake, numbers have patterns as well.”. I noticed now that she was loosing her concentration and said to her: “we are almost done, but before that, could you please add $1+2+3+\cdots+20$ in the same way as you just did?”. She answered quickly: “five times twenty-one that makes hundred and five”. I asked: “why?”. She said: “you see one plus twenty is twenty-one and two plus nineteen is twenty-one and so on, and there are five twenty-ones, right?”. I said to her: “I can see where twenty-one comes from and I could not see where five comes from, could it be possible that five comes from the task you just finished?”. She thought for a while and smiled embarrassingly and said: “ten twenty-ones that make two hundred and ten”.

I was somehow not surprised by the mistake she made at the end since I firmly believe that a transition from one structure to another is a long and thorny process. Anyway, before the girl left me, I asked her to think about the following pattern when she has time:

\[
\begin{array}{cccccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \bullet \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \bullet & \bullet \\
\circ & \circ & \circ & \circ & \circ & \circ & \bullet & \bullet & \bullet \\
\circ & \circ & \circ & \circ & \circ & \bullet & \bullet & \bullet & \bullet \\
\circ & \circ & \circ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\circ & \circ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\circ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

which is the graphic dialectic of

\[
2 \cdot (1 + 2 + 3 + \cdots + 10) = 10 \cdot 11.
\]

Sometimes I wander about how that girl reacts if the sum is $1+2+\cdots+11$ before she was exposed to $1+2+\cdots+10$. The fact that eleven numbers can not be organized by pairing leads to imperfection which is more psychological than mathematical. I did not experiment this thought on anyone and shall not do it in the future either, since I believe that structure of this new
sum should be centered around the structure just discussed, not other way around. It is very easy for the child to come up with thing like

\[1 + 2 + \cdots + 10 + 11 = (1 + 2 + \cdots + 10) + 11 = 55 + 11 = 66;\]

or even better, the pattern rooted in the picture above tells us to add \(s = 1 + 2 + 3 + \cdots + 10\) with itself and the result is \(2s = 10 \cdot 11\) so that \(s = 10 \cdot 11 / 2\).

Now what about \(s = 1 + 2 + 3 + \cdots + n\)? The discussion made so far should give a clear idea of how to proceed. Either we draw a similar picture or else we proceed as follows. We have

\[2s = s + s = (1 + 2 + 3 + \cdots + n) + (n + (n - 1) + (n - 2) + \cdots + 1)\]

\[= (n + 1) + (n + 1) + (n + 1) + \cdots + (n + 1) = n(n + 1),\]

so that \(s = n(n + 1)/2\) as we promised before.

The example of adding together all natural numbers between one and one hundred has been used many times and shall be used many times in mathematics education. Why we take the example or what is the purpose of taking it? It is much difficult to answer and there must be many different answers depending who answer. I used this example in my classes at times and reactions were more or less the same. No one seems find the example difficult to understand and many were impressed by the simplicity and the beauty of the reasoning used by Gauss, and at the same time the students doubted about that they can come up that reasoning themselves. This is no surprise, the students add in a normal fashion while Gauss added in a revolutionary fashion, they have different first principles.

No matter how the example is judged within mathematics education, one thing is clear, it is entertaining. From the perspective of pure mathematics it is also an interesting example, for it supplies the natural numbers with a local structure at the beginning and it offers a testing criterion for the structure at the end. Thus the structure is \(1 + 2 + \cdots + n\) and the ground for testing whether a given number \(m\) belongs to the structure is the equation \(n(n + 1)/2 = m\).

At the moment when major premises have been formulated for a structure, the structure ceases from being; a process of re-structuring starts. Structure of production in economy is steered by a market and theoretical structure in physics is regulated by phenomena. Structure in mathematics invokes intuition, methods as well as phenomena. In a process of re-structuring \(1 + 2 + \cdots + n\), we can proceed at least in the following ways. One structure can be

\[1 + 3 + 5 + 7 + \cdots + (2n - 1),\]

another can be

\[2 + 4 + 6 + 8 + \cdots + 2n,\]
or
\[1^2 + 2^2 + 3^2 + \cdots + n^2,\]
or
\[1^3 + 2^3 + 3^3 + \cdots + n^3,\]
or in general
\[1^k + 2^k + 3^k + \cdots + n^k.\]

Being structures on natural numbers, the principal issue is still that whether a given number belongs to the structures. Since new structures are intuitive continuation of the old one \(1 + 2 + \cdots + n\), in an inquiry into the principal issue, there are three ways to proceed.

- The first way is to make use the already made results, or in other words, to make a continuation upon the old structure.
- The second way is to make a parallel continuation whenever it is possible.
- The third way is regulate the old structure by the new structures in such fashion that the first way or the second way can be implemented.

The first way is applicable to the first and the second new structures. In fact it holds
\[2 + 4 + 6 + 8 + \cdots + 2n = 2 \cdot (1 + 2 + 3 + 4 + \cdots + n) = 2 \cdot \frac{n(n + 1)}{2} = n(n + 1),\]
and
\[1 + 3 + 5 + \cdots + (2n - 1) = (1 + 2 + 3 + \cdots + 2n) - (2 + 4 + 6 + \cdots + 2n)\]
\[= \frac{2n(2n + 1)}{2} - n(n + 1) = n^2.\]
For these two structures the second way is applicable as well. Indeed
\[\left(1 + 3 + 5 + \cdots + (2n - 1)\right) + \left((2n - 1) + (2n - 3) + (2n - 5) + \cdots + 1\right)\]
\[= \underbrace{2n + 2n + 2n + \cdots + 2n}_{n} = 2n^2\]
and
\[\left(2 + 4 + 6 + \cdots + 2n\right) + \left(2n + (2n - 2) + (2n - 4) + \cdots + 2\right)\]
\[= \underbrace{(2n + 2) + (2n + 2) + (2n + 2) + \cdots + (2n + 2)}_{n} = n \cdot (2n + 2),\]
so that
\[1 + 3 + 5 + \cdots + (2n - 1) = n^2, \quad 2 + 4 + 6 + \cdots + 2n = n(n + 1).\]
However both the first way and the second way are not applicable in other structures. Take $1^2 + 2^2 + 3^2 + 4^2$ as an example. We find immediately that the old way does not work, for $1^2 + 4^2 = 17$ and $2^2 + 3^2 = 13$ but $17 \neq 13$. We certainly can not blame Gauss for the insufficiency, in fact if there is anyone to blame then it must be ourselves, for it is we who chose to look at the new structures from the old perspective. So realized, we must wander about whether there are other perspectives of which we can look at both the old structure and the new structures.

So far we have only dealt with a simple sum $1 + 2 + \cdots + n$ and have used a very special way to make additions in the sum. In searching for new perspectives it seems reasonable to delve into such sums to a greater extent.

The sums we have dealt with are special cases of a general sum

\[ a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k. \]

The letter $k$ is insignificant, one can write the same series as $\sum_{j=1}^{n} a_j$ or as $\sum_{p=1}^{n} a_p$. In such a sum there are many additions to be executed, in fact so many that finding general pattern is hopeless unless sums are very special.

The sum $a + a + \cdots + a$ is no doubt a special one. Indeed we have

\[ \sum_{k=1}^{n} a = na. \]

Among all sums of two terms the simplest is the law of cancellation meaning

\[ a + (-a) = 0. \]

The number $a$ can be very complicated and very large, but the sum $a + (-a)$ is always zero.

For the sake of argument let us do a very simple computation by making use of the law of cancellation. We have

\[ \sum_{k=1}^{n} 1 = \sum_{k=1}^{n} ((k+1) - k) \]

\[ = (2 - 1) + (3 - 2) + (4 - 3) + \cdots + ((n+1) - n) = (n+1) - 1 = n, \]

since all numbers except for 1 and $n+1$ appear in pair with opposite signs and since they disappear in the process of summation. This simple observation creates in fact a way of computing the sums. To make sure this is the case let us do another computation. We want to compute

\[ \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)}. \]
Since \[ \frac{1}{k(k+1)} = \frac{(k+1) - k}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}, \]
it follows that \( \sum_{k=1}^{n} \frac{1}{k(k+1)} \) is equal to \[
\sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) = \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \cdots + \left( \frac{1}{n} - \frac{1}{n+1} \right)
\]
\[= \frac{1}{1} - \frac{1}{n+1} = 1 - \frac{1}{n+1} = \frac{n}{n+1}. \]

The use of the cancellation in summation, like the case just showed, should be exemplified; for it is one of simplest methods in controlling a sum with a large amount of summands. Of course it is not the only one method but a significant one which together with some others transform seemingly unmanageable into reasonably manageable. The law of cancellation can appear in another form in removing the unmanageable. Some methods in mathematics are disguised forms of one and the same method. Another form of cancellation appears in managing the geometric sums.

A finite geometric sum determined by a number \( q \) is \( 1 + q + q^2 + q^3 + \cdots + q^n \). Now we can proceed in at least two ways. Let \( s = 1 + q + q^2 + q^3 + \cdots + q^n \). Multiplying the equation by the number \(-q\) we get \( -qs = -q - q^2 - q^3 - \cdots - q^{n+1} \).

It should be clear now in which way the law of cancellation comes in. Adding the preceding equations we get \( s - qs = 1 - q^{n+1} \implies s = \frac{1 - q^{n+1}}{1 - q} \).

The other way is the use of the law of cancellation in a disguised form. We write \[ s = 1 + q + q^2 + q^3 + \cdots + q^n = 1 + q \cdot (1 + q + q^2 + \cdots + q^{n-1}) \]
\[= 1 + q \cdot (1 + q + q^2 + \cdots + q^{n-1} + q^n - q^n) = 1 + q \cdot (s - q^n), \]
so that \( s - qs = 1 - q^{n+1} \implies s = \frac{1 - q^{n+1}}{1 - q} \).

Needless to say the reasoning is valid under the assumption that \( q \neq 1 \). If \( q = 1 \) on the other hand then \( 1 + q + q^2 + \cdots + q^n = n + 1 \), so that we can summarize the results obtained as

\[1 + q + q^2 + \cdots + q^n = \begin{cases} \frac{1 - q^{n+1}}{1 - q}, & q \neq 1, \\ n + 1, & q = 1. \end{cases}\]
A typical feature of mathematics education is the art of using mathematics instruction at some educational singular points. In fact the use is so decisive that mathematics education can not proceed any more if otherwise. Mathematics education at those singular points is one of few greatest challenges for mathematics teachers. I myself do not think of I have succeeded to my full satisfaction at the singular points. I shall use the example I began with earlier in the discussion.

Remember that we would like to look at the sum $s_1 = 1 + 2 + 3 + \cdots + n$ again with a hope that it will give us some help in dealing with another sum $s_2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$. And this time we shall try to make use of the law of cancellation. I, as a mathematics teacher, know from my earlier experience that I should start with

$$(k + 1)^2 = k^2 + 2k + 1$$

which is a new first principle. Then we write it as

$$2k = (k + 1)^2 - k^2 - 1.$$ 

Since

$$s_1 = \sum_{k=1}^n k$$

then we arrive at a reasoning such as

$$2s_1 = \sum_{k=1}^n 2k = \sum_{k=1}^n ((k + 1)^2 - k^2 - 1)$$

$$= (2^2 - 1^2 - 1) + (3^2 - 2^2 - 1) + (4^2 - 3^2 - 1) + \cdots + ((n + 1)^2 - n^2 - 1)$$

$$= -1^2 + (n + 1)^2 - n = (n + 1)^2 - (n + 1) = (n + 1)(n + 1 - 1) = (n + 1)n,$$

so that

$$s_1 = 1 + 2 + 3 + \cdots + n = \frac{(n + 1)n}{2}.$$ 

The result was gotten once before with another first principle. The new first principle $(k + 1)^2 = k^2 + 2k + 1$ is the third row in Pascal’s triangle. The connection between the sum $s_1$ and the third row in Pascal’s triangle is definitely an educational singular point, for once this point is acquired, a new horizon is opened. It is now natural to inquire into a connection between $s_2$ and the fourth row in Pascal’s triangle which is

$$(k + 1)^3 = k^3 + 3k^2 + 3k + 1.$$ 

Change it into

$$3k^2 = (k + 1)^3 - k^3 - 3k - 1.$$
Write
\[3s_2 = 3 \sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} 3k^2 = \sum_{k=1}^{n} ((k+1)^3 - k^3 - 3k - 1)\]
\[= \left(2^3 - 1^3 - 3 \cdot 1 - 1\right) + \left(3^3 - 2^3 - 3 \cdot 2 - 1\right) + \left(4^3 - 3^3 - 3 \cdot 3 - 1\right) + \cdots + \left((n+1)^3 - n^3 - 3 \cdot n - 1\right)\]
\[= -1^3 + (n+1)^3 - 3 \cdot (1 + 2 + 3 + \cdots + n) - n = -1 + (n+1)^3 - 3s_1 - n\]
\[= (n+1)^3 - \frac{3n(n+1)}{2} - (n+1) = (n+1) \cdot \left((n+1)^2 - \frac{3n}{2} - 1\right)\]
\[= (n+1) \cdot \frac{2(n^2 + 2n + 1) - 3n - 2}{2} = (n+1) \cdot \frac{2n^2 + n}{2} = (n+1)n(n+1/2),\]
so that
\[s_2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(n+1/2)}{3} = \frac{n(n+1)(2n+1)}{6}.\]

In order to feel safe psychologically one can take the fifth row in Pascal’s triangle and try to make a connection to \(1^3 + 2^3 + \cdots + n^3\). The fifth row is
\[(k+1)^4 = k^4 + 4k^3 + 6k^2 + 4k + 1.\]
By modifying what have been done slightly it follows that
\[s_3 = \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.\]

With the work done so far we can induce a methodological structure. Let
\[s_m = \sum_{k=1}^{n} k^m = 1^m + 2^m + 3^m + \cdots + n^m.\]
All numbers \(s_m\) can be constructed by the Pascal’s triangle. Indeed the number \(s_m\) can be constructed directly upon the \((m+1)\)-th row in that triangle and the numbers \(s_1, s_2, \ldots, s_{m-1}\). This methodological structure is good in mathematics but bad in practical computation. Say we want compute \(s_{100}\). In order to do the task we need successively to compute \(s_1, s_2, \ldots, s_{99}\). The work involved is overwhelming and practically impossible.

Now that the methodological structure is well established, and at the same time de-structure is also accomplished, and then re-structure starts. In order to organize the findings and to find out a structure on them, we observe that all \(s_1, s_2, s_3\) are polynomials and
\[s_1 = \sum_{k=1}^{n} k = \frac{n^2}{2} + \cdots, \quad s_2 = \sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \cdots, \quad s_3 = \sum_{k=1}^{n} k^3 = \frac{n^4}{4} + \cdots\]
where “…” denotes the lower order terms. All these lead naturally to an intuitive structure

\[ s_m = \sum_{k=1}^{n} k^m = \frac{n^{m+1}}{m+1} + \cdots \]

where “…” is as above the lower order terms. In such writing the lower order terms exist but are insignificant. Mathematically it does not matter if they have a simple or complex expression, they are not meant to be expressed clearly, for the structure in searching is independent upon them. It is even better that the lower order terms do not appear explicitly, and a good mathematical form of this intuitive structure is thereby

\[ \lim_{n \to +\infty} \frac{m+1}{n^{m+1}} \cdot \sum_{k=1}^{n} k^m = 1. \]

The first principle, using Pascal’s triangle, in dealing with \( s_m \) is of theoretical value but not practical. However, if we drop the exact information on \( s_m \) out, instead of searching for approximate information, then much can be done. For this we need to search for a new first principle.

We start with treating the isolated summands \( 1^m, 2^m, 3^m, \ldots, n^m \) in \( s_m \) as one piece, and noticing that they are special values of a single function \( x^m \). Since its derivative is \( mx^{m-1} \), the function \( x^m \) increases strictly so long as \( m > 0 \). Let us draw a picture

![Diagram](image)

And let us interpret \( k^m \) as the area of a rectangle of base 1 and height \( k^m \). By comparing the areas in the picture, we have

\[ \int_{k-1}^{k} x^m \, dx < k^m < \int_{k}^{k+1} x^m \, dx, \]

so that

\[ \frac{m+1}{n^{m+1}} \cdot \sum_{k=1}^{n} \int_{k-1}^{k} x^m \, dx < \frac{m+1}{n^{m+1}} \cdot \sum_{k=1}^{n} k^m < \frac{m+1}{n^{m+1}} \cdot \sum_{k=1}^{n} \int_{k}^{k+1} x^m \, dx \]
which is
\[
\frac{m+1}{n^{m+1}} \cdot \int_0^n x^m \, dx < \frac{m+1}{n^{m+1}} \cdot \sum_{k=1}^n k^m < \frac{m+1}{n^{m+1}} \cdot \int_1^{n+1} x^m \, dx.
\]

Since
\[
\int_a^b x^m \, dx = \left[ \frac{x^{m+1}}{m+1} \right]_a^b = \frac{b^{m+1} - a^{m+1}}{m+1},
\]
it follows that
\[
\frac{m+1}{n^{m+1}} \cdot \int_0^n x^m \, dx = \frac{m+1}{n^{m+1}} \cdot \frac{n^{m+1}}{m+1} = 1
\]
and
\[
\frac{m+1}{n^{m+1}} \cdot \int_1^{n+1} x^m \, dx = \frac{m+1}{n^{m+1}} \cdot \frac{(n+1)^{m+1} - 1}{m+1} = \frac{(n+1)^{m+1} - 1}{n^{m+1}}.
\]
Together we have
\[
1 < \frac{m+1}{n^{m+1}} \cdot \sum_{k=1}^n k^m < \frac{(n+1)^{m+1} - 1}{n^{m+1}} \to 1
\]
as \( n \to +\infty \), so that
\[
\lim_{n \to +\infty} \frac{m+1}{n^{m+1}} \cdot \sum_{k=1}^n k^m = 1,
\]
or
\[
\sum_{k=1}^n k^m \sim \frac{n^{m+1}}{m+1}.
\]

I shall take this structure as my final destination of my story, though the mathematical story can continue. In fact the story can lead us to the highest part of mathematics, for example we can look \( z \) upon as a complex number and \( n \) as \( +\infty \) in which case the mathematical object is the Riemann’s zeta function
\[
\zeta(z) = \sum_{k=1}^{+\infty} k^{-z}.
\]

### 6.6 Numbers in Minimum Structure

The title of this section could have been *The Method of Mathematical Induction*. The reason is that the method of mathematical induction is one of the most important applications of the minimum structure which holds for many classes of numbers including the class of natural numbers and is in addition popular in mathematics education.
The method of mathematical induction gives rise to a great challenge in mathematics education which manifests itself in passing from finite to infinite. The challenge is psychological and can be reduced by invoking a minimum structure of natural numbers. This structure facilitates passing from infinite back to finite and therefore has its educational strength. The method of mathematical induction as an educational task should not be taken merely as a method of verifying a given judgment, but a method of conjecturing, refuting and confirming.

In the first half of the 1960s a book-series appeared in China, the name of the series is *Non-Curriculum Mathematical Readings for Pupils of High School*. Some lines in the general preface written by the editors are very interesting from didactical perspective, thereby I shall translate them here.

Mathematics, at high schools, is an instrumental science on a basic level. The aim of teaching and learning mathematics must be the one that pupils manipulate mathematics as a tool, so as to a solid foundation is laid for their future participation in production and for their future studies. To facilitate studying, besides quality of teaching must be lifted at maximum, a cooperation among various participants is needed. The aim of this book-series is to assist teaching and learning of mathematics at high schools in the following areas, in strengthening mastering of the basic knowledge, in lifting the basic technique to a higher level, in widening the horizon, in cultivating the pupils’ interesting for mathematics, and after all in facilitating the pupils’ future participation in production and future studies.

Editors, *Non-Curriculum Mathematical Readings for Pupils of High School*

One book in the series was written by Chinese mathematician Loo-Keng Hua. The name of the book is *The Method of Mathematical Induction*, in which he wrote: “When I studied the method of mathematical induction at high school, I thought at first I mastered the method. However, later, more I thought, more I felt dissatisfaction, for I felt something is missing.”. Hua used the following example to show his dissatisfaction.

**Example 6.6.1.** To show that

\[1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n + 1)^2}{4}\]

holds for \(n = 1, 2, 3, \cdots\).

**Proof.** In the proof I shall proceed in a very standard fashion.

Firstly, I shall check if the formula to be proved is true at \(n = 1\). In this case the formula becomes

\[1^3 = \frac{1^2(1 + 1)^2}{4}\]

which is clearly true.
Secondly, assume that for some natural number \( k \) the formula to be proved is true at \( n = k \), namely, assume that

\[
1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}
\]

is true. Adding \((k+1)^3\) on the both sides of this true formula we obtain

\[
1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3.
\]

Since the right side is equal to

\[
(k+1)^2 \cdot \frac{k^2 + 4(k+1)}{4} = (k+1)^2 \cdot \frac{k^2 + 4k + 2^2}{4} = \frac{(k+1)^2(k+2)^2}{4}
\]

then

\[
1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}
\]

which shows that the formula to be proved is true at \( n = k + 1 \).

By combining these two steps together we conclude that the formula to be proved is true for all natural numbers \( n \).

The formula and the proof are all correct. As to the formula itself, Hua felt something missing, where did the formula come from? As to the proof the student often feels something mysterious. The feeling depends upon the standard presentation of the method in the context of mathematics education, it consists of a statement to be proved together with two steps. At the beginning the teacher probably explains to the student the reason on which the two steps prove the statement. The explanation is probably metaphorical based on some cognate things such as domino. And then the two steps take completely over and become an algorithm. The method of mathematical induction is a powerful weapon in the realm of mathematics, it is one of very few methods which can be used in almost all disciplines within mathematics. Thus; it can be used in Algebra, Geometry as well as Analysis; in fact, in some disciplines such as Combinatorics and Theory of Graphs, it is the most important method. The method is powerful for it lies in the unification of conjecture and refutation, indeed it is a synthesis of them. Base on observations, conjectures are established, and are verified or refuted by the method, new conjectures are created and so on.

For a further discussion I need to formalize the method. Object in the method of mathematical induction is, in usual cases, an infinite sequence of judgments

\[
P_1, \quad P_2, \quad P_3, \quad \cdots,
\]

and premises of the method are propositions \( M_1 \) and \( M_2 \). The proposition \( M_1 \) is the one saying that the judgment \( P_1 \) is in truth, and the proposition
$M_2$ is the other saying that $P_{k+1}$ is in truth if $P_k$ is in truth, where $k$ is some natural number. Conclusion of the method is proposition which says that all of $P_1, P_2, P_3, \cdots$ are in truth. The method of mathematical induction is thus

- $M_1$: $P_1$ is true,
- $M_2$: $P_{k+1}$ is true if $P_k$ is true for some $k$,
- conclusion: $P_n$ is true for all $n$.

We have apparently the following syllogism, for when $M_1$ and $M_2$ are taken as a thesis and an antithesis, then the conclusion is a synthesis of them,

$$
P_n \text{ are true for all } n
$$

$P_1 \text{ is true}$ $P_{k+1} \text{ is true if } P_k \text{ is true}$

The method of mathematical induction concerns itself not judgments as such but propagation of truth of judgments. Our discussion on the method of mathematical induction consists of three parts — that on objects, that on premises and that on conclusion.

An object is a mixture of observed phenomena and conjectures, and the former are in truth while the latter can be in truth or in falsity.

Example 6.6.2. Let $P_1$ be the judgment that Sun comes out from east at day one, $P_2$ be the judgment that Sun comes out from east at day two, $P_3$ be the judgment that Sun comes out from east at day three, and so on.

Example 6.6.3. Let $P_1$ be the judgment that the number $1^2 + 1 + 17$ is prime, $P_2$ be the judgment that the number $2^2 + 2 + 17$ is prime, and in general $P_n$ be the judgment that the number $n^2 + n + 17$ is prime.

Example 6.6.4. Let $P_0$ be the judgment that the number $2^{20}$ is prime, $P_1$ be the judgment that the number $2^{21}$ is prime, $P_2$ be the judgment that the number $2^{22}$ is prime, and in general $P_n$ be the judgment that the number $2^{2n}$ is prime.

Example 6.6.5. Let $P_1$ be the judgment that

$$1 = \frac{1 \cdot (1+1)}{2},$$

$P_2$ be the judgment that

$$1 + 2 = \frac{2 \cdot (2+1)}{2},$$

and in general $P_n$ be the judgment that

$$1 + 2 + \cdots + n = \frac{n \cdot (n+1)}{2}.$$
For the moment a pattern of truth of these examples is the same. A few of the first judgments are in truth by observing, and the form of other judgments is induced by the form of those few true judgments. This reasoning is one of the most fundamental reasonings in entire epistemology, and is called induction. Judgments whose truth to be determined are conjectures, so that an object of the method of mathematical induction is a sequence of judgments whose section at the very beginning of the sequence consists of true judgments and the rest consists of conjectures.

Two premises are establishment of truth of two judgments $P_1$ and $P_{k+1}$ of very different kinds. The truth of $P_1$ is absolute while the truth of $P_{k+1}$ is conditional. The absolute truth of $P_1$ is attained by verification, therefore the premise $M_1$ is a simple matter and seldom causes confusion in mathematics education. Contrast to this, the premise $M_2$ is a great challenge in mathematics education, the challenge is not so much mathematical but indeed psychological. At the bottom it is a battle between appearances of two formulations:

- to assume that $P_k$ is true for some $k$,
- to prove that $P_n$ is true for all $n$.

A focus of attention is thereby on three pairs:

- to assume and to prove,
- for some $k$ and for all $n$,
- $P_k$ and $P_n$.

Let us think of the three issues from the perspective of the student. First of all, is there any difference between $P_k$ and $P_n$? The student can find plenty of cases in his memory that can lead him to believe that there is no difference between these two. For instance

$$
\int_a^b f(s) \, ds = \int_a^b f(t) \, dt,
$$

$$
\sum_{k=1}^m a_k = \sum_{n=1}^m a_n,
$$

and the same function can be written sometimes as $f(x)$ and sometimes as $f(u)$. And then, the whole task is to prove that $P_n$ is true, isn’t it? Why in the whole world we are assuming that $P_k$ is true? In that case, can the student simply prove that $P_n$ is true by assuming that $P_n$ is true? At last, what $k$ and $n$ have anything to do with $P$? In one of the preceding examples, $P$ stands for the statement that Sun comes out from east while $k$ is a name for a day. Mathematicians do not need to care for these questions,
for they are not mathematical issues in the first place but psychological ones. However mathematics teachers on the other side must take the questions as their teaching premises, and must combine the psychological issues with the mathematical forms together and turn them into different perspectives.

Quantifiers in logic should be taught before the method of mathematical induction is introduced. In fact the sooner the better. Mathematics education gains nothing by delaying. Two quantifiers are particular important, and they are \(\text{for some } n\) and \(\text{for all } n\). The formulations of these two quantifiers are used consistently throughout mathematics and should not be treated separately in mathematics education. The quantifier, for some \(n\), is a special case of the quantifier, for all \(n\); so that they lie in a synthesis of particularity and generality. The quantifiers can only be used in a situation of asserting say \(P_n\). Thus we assert that \(P_n\) is true for some \(n\) or else that \(P_n\) is true for all \(n\). The complex difference is hidden by the simple formulation of the quantifiers. That difference enters on the stage if the logic addition \(\cup\) and the logic multiplication \(\cap\) are introduced, in which case, the assertions

- \(P_n\) is true for some \(n\),
- \(P_n\) is true for all \(n\)

become

- \(\cup_n P_n\) is true,
- \(\cap_n P_n\) is true.

Hence by the quantifiers, essence of the method of mathematical induction finds its place in a synthesis of particularity and generality.

A conclusion here does not mean truth of a thing but that the two steps together implies the truth of that thing. It is a part of the method of mathematical induction — the third step, but remains hidden most of time. This step can be achieved mathematically as well as didactically. A mathematical enunciation is seldom picked up in mathematics education, and a didactical one is mentioned at times. It is precisely the didactical treatment that paints this hidden step in esoteric. Said so, let us now have a look at the didactical enunciation of the third step. In fact, it is very short. The first step simply says that \(P_1\) is true. Since now \(P_1\) is true it follows from the second step that \(P_2\) is true, and since now \(P_2\) is true it follows again from the second step that \(P_3\) is true, and so on. Continuing in this way, we conclude that all \(P_n\) are true.

The treatment of the third step given above is one of perspectives. Anytime when we feel a perspective unsafe or esoteric, it is then the time for us to change to another perspective.

The set \(\mathbb{N}\) of all natural numbers has an important structure which, seldom mentioned in mathematics education, this structure has a far-reaching
influence in the development of mathematics. I call this structure minimum structure of natural numbers. The minimum structure is a claim that any non-empty subset of \( \mathbb{N} \) contains the least number (the minimum). Thus let \( E \) be a non-empty subset of \( \mathbb{N} \), it follows of the minimum structure that there exists a number \( m \in E \) such that \( m \leq x \) for all \( x \in E \). First of all let us show that the minimum is unique, which justifies a usage the minimum. To this end assume that \( m_1 \) and \( m_2 \) are both the least numbers. It follows from the fact that \( m_1 \) is the least number that \( m_1 \leq m_2 \) because of \( m_2 \in E \), and it follows from the fact that \( m_2 \) is the least number that \( m_2 \leq m_1 \) because of \( m_1 \in E \). Together we have \( m_1 = m_2 \). This is the uniqueness of the minimum. The existence of the minimum can be rigorously proved in the following fashion. Since \( E \) is not empty, there exists at least number \( n_1 \in E \). If the number \( n_1 \) is the minimum then the claim is proved. If \( n_1 \) is not the minimum then there exists a number \( n_2 \in E \) with \( n_2 < n_1 \). Now we can iterate the preceding argument. If the number \( n_2 \) is the minimum then the claim is proved. If \( n_2 \) is not the minimum then there exists a number \( n_3 \in E \) with \( n_3 < n_2 < n_1 \). Since there are at most \( n_1 \) natural numbers which are less than \( n_1 \), then after at most \( n_1 \) times iterations we definitely get the minimum.

With the minimum structure we can put the third step of the method of mathematical induction on a solid mathematical foundation. We proceed as follows. For a given sequence of judgments \( P_1, P_2, \cdots \), we know some of them are true and other are not. Consequently it is always interesting to inquiry into the density of truth, namely, to study the so-called the set of truth. Thus, in our case, let \( T \) be the set of all natural number \( m \) with which \( P_m \) is true. Of course all natural numbers which are not in \( T \) constitute the set \( F \) of falsity. After having accomplished the first step and the second step in the method of mathematical induction, let us show that \( P_n \) is true for all \( n = 1, 2, 3, \cdots \). This amounts to the same thing as showing \( T = \mathbb{N} \). We shall show this by showing that the negation is impossible. To this end assume that \( T \neq \mathbb{N} \). Since the set \( T \) is a subset of \( \mathbb{N} \), the assumption means that \( F \) is not empty. Since now \( F \) is a non-empty subset of \( \mathbb{N} \), it follows from the minimum structure of natural numbers that \( F \) contains the least number \( h \). Because of \( h \in F \), we assert \( P_h \) is false. Since \( P_h \) is false while \( P_1 \) is true (the first step), it follows that \( h > 1 \). Since \( h \) is the least number in \( F \) then \( h - 1 \) is not in \( F \) thus in \( T \). Let \( k = h - 1 \). Since \( P_k \) is true then it follows from the second step that \( P_{k+1} \) is also true. Since \( h = k + 1 \) then \( P_h \) is true. Put them together, this is what we have proved. Under the assumption that \( T \neq \mathbb{N} \), we proved that \( P_h \) is false and that \( P_h \) is true for some \( h \). The mathematical universe we live in is an ordered universe with orders given by logic. One of the oldest order was explicitly spelled out by Aristotle under the name law of contradiction. Since then this law has been being accepted as one of the most important laws in entire mathematics. It is this law that conserves the truth under a valid mathematical reasoning.
The law of contradiction says that, for each mathematical object $A$, there is a well-defined negation $\neg A$, such that

- $\neg(\neg A) = A$,
- either $A$ is in truth or else $\neg A$ is in truth,
- if $A$ is in truth then the negation $\neg A$ is in falsity,
- if $A$ is in falsity then the negation $\neg A$ is in truth,
- if a valid mathematical reasoning leads $A$ to a contradiction then $A$ is in falsity.

Here the contradiction means, for some $A$, the $A$ is in truth and in falsity at same time. With the help of the law of contradiction, let us continue with what we left. Since the assumption $T \neq \mathbb{N}$ leads by a valid mathematical reasoning to a contradiction ($P_h$ is false and $P_h$ is true), it follows from the law of contradiction that $T \neq \mathbb{N}$ is in falsity, thereby the negation of $T \neq \mathbb{N}$ is in truth, that is $T = \mathbb{N}$ is in truth. The truth of $T = \mathbb{N}$ is the same as that $P_h$ is true for all $n \in \mathbb{N}$. The third step is thereby confirmed.

The approach based upon the minimum structure eliminates the psychological insecurity caused by going 1, 2, 3, ··· all the way up to the infinity by going down to the minimum which is a finite number.

In mathematics education the method of mathematical induction often gives rise to another perspective looking at the things already done. Even for mathematicians this is important. After a long calculation a result is reached. In order to eliminate possible human errors made in the process of calculation, it always feels safe to check the result by the method of mathematical induction. For mathematics teachers it generates good problems. For instance, combining what have been done earlier, we can set up many problems.

**Example 6.6.6.** To show by the method of mathematical induction that

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

holds for $n = 1, 2, 3, \cdots$.

**Example 6.6.7.** To show by the method of mathematical induction that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

holds for $n = 1, 2, 3, \cdots$. 
Example 6.6.8. To show by the method of mathematical induction that

\[ \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \]

holds for \( n = 1, 2, 3, \cdots \).

The two steps \( M_1 \) and \( M_2 \) in the method of mathematical induction are absolutely necessary for a valid mathematical induction. For instance the method of mathematical induction can not prove

\[ \sum_{k=1}^{n} k^2 = \frac{n(n+1)}{2} \]

though it is true at \( n = 1 \), for it is false for all \( n > 1 \). And the method of mathematical induction can not prove either

\[ n = n + 1 \]

though \( M_2 \) is true. Indeed, assuming that \( k = k + 1 \) for some \( k \) then \( k + 1 = k + 1 + 1 \) by adding 1 on the both sides. If we would have concluded \( n = n + 1 \) for all natural numbers without checking the step \( M_1 \) then we would have proved that

\[ 1 = 2 = 3 = 4 = \cdots \]

Upon checking the first step, we have \( 1 = 2 \) which is false.

The method of mathematical induction itself can not produce new knowledge, but it plays a very decisive role in verifying new knowledge produced by induction. Knowledge produced by induction can be true or false.

Upon checking numbers of the form \( n^2 + n + 17 \) for \( n = 1, 2, \cdots, 15 \) we find out that all of them are prime numbers, and thereby induce that all number of that form are prime. However when \( n = 16 \) the number of that form is \( 16^2 + 16 + 17 = 289 = 17^2 \) which is not prime. Mathematicians can show that numbers of the form \( n^2 + n + 72491 \) are all prime when \( n = 1, 2, 3, \cdots, 11000 \). It seems very safe to induce that all numbers of this form are prime. But the number of this form is not prime when \( n = 72490 \). In fact, the numbers of the form \( n^2 + n + m \) contain many prime numbers but also composite numbers, for it is always a composite number when \( n = m - 1 \). Indeed, in that case we have

\[ n^2 + n + m = (m - 1)^2 + (m - 1) + m = m^2 - 2m + 1 + 2m - 1 = m^2. \]

Now we have a famous example of this sort. Fermat computed the numbers of the form

\[ 2^{2^n} + 1 \]
for \( n = 0, 1, 2, 3, 4 \), and the results are 3, 5, 17, 257, 65537 which are prime; thereby he claimed that all numbers of this form are prime. However Euler computed the number of this form when \( n = 5 \) and found
\[
2^{2^5} + 1 = 4294967297 = 641 \cdot 6700417
\]
which is not prime.

So far in our treatment of the method of mathematical induction, the sequence of judgments \( P_1, P_2, \cdots \) begins at 1. The method of mathematical induction works equally well for other sequences such as
\[
P_m, P_{m+1}, P_{m+2}, \cdots
\]
which begins at \( m \). In this case, \( M_1 \) and \( M_2 \) are changed into
- \( M_1^* : P_m \) is true,
- \( M_2^* : \) the same as \( M_2 \).

**Example 6.6.9.** To show that
\[
2^n > n^2
\]
for all \( n = 5, 6, 7, \cdots \).

*Proof.* When \( n = 5 \) we have
\[
2^5 = 32 > 25 = 5^2
\]
so that the judgment is true at \( n = 5 \).

Assume now that, for a natural number \( k \) which is greater than 5, \( 2^k > k^2 \). Let us show that
\[
2^{k+1} > (k + 1)^2.
\]
We have
\[
2^{k+1} = 2 \cdot 2^k > 2 \cdot k^2 > (k + 1)^2
\]
for
\[
2 \cdot k^2 - (k + 1)^2 = k^2 - 2k - 1 = (k - 1)^2 - 2 > 7 > 0.
\]
It follows from the method of mathematical induction that \( 2^n > n^2 \) for all \( n > 4 \). \( \square \)

Now we know that the method of mathematical induction presented above is base on the minimum structure of natural numbers. We can make some variants of the method, and sometimes it is the variants that work not the standard one.

The first variant is
- \( M_1^{**} : \) the same as \( M_1 \),
• $M_2^{**}$: $P_{k+1}$ is true if all $P_1, P_2, \ldots, P_k$ are true.

The second variant is

• $M'_1$: $P_1$ and $P_2$ are true,

• $M'_2$: $P_{k+1}$ is true if both $P_k, P_{k-1}$ are true.

The third variant, in cognation to the second, is

• $M''_1$: $P_1, P_2$ and $P_3$ are true,

• $M''_2$: $P_{k+1}$ is true if all $P_k, P_{k-1}, P_{k-2}$ are true.

The fourth variant is

• $M'''_1$: $P_1$ and $P_2$ are true,

• $M'''_2$: $P_{k+1}$ is true if $P_{k-1}$ is true.

In all four variants $k$ is some natural number, and a justification of all variants is the same as before and is based on the minimum structure of natural numbers.

We now use the first variant to prove an important theorem in Theory of Numbers.

**Example 6.6.10.** Every natural number which is greater than one can be written as a product of prime numbers.

**Proof.** Since the number 2 is prime then $2 = 2$ is the product.

Assume now that each number of 2, 3, $\ldots, k$ is a product of prime numbers. We shall show that the number $k + 1$ is also such a product. With respect to $k + 1$ we have only two options, either it is prime or else composite. If $k + 1$ is prime then $k + 1 = k + 1$ is the product. If on the other hand $k + 1$ is composite then $k + 1 = k_1 \cdot k_2$ where both $k_1$ and $k_2$ are some natural numbers between 2 and $k$. According to the assumption made above $k_1 = p_1 p_2 \cdots p_s$ and $k_2 = q_1 q_2 \cdots q_t$ where $p_1, \cdots, p_s$ and $q_1, \cdots, q_t$ are all prime numbers. Then

$$k + 1 = p_1 p_2 \cdots p_s q_1 q_2 \cdots q_t$$

is the product of prime numbers.

It follows from the method of mathematical induction that every natural number which is greater than one can be written as a product of prime numbers.

Let us to make use of the second variant in proving
Example 6.6.11. Let $a_1, a_2, a_3, \cdots$ be a sequence of numbers satisfying
\[ a_{n+1} = 3a_n - 2a_{n-1}, \quad a_1 = 3, \quad a_2 = 5. \]

To show that
\[ a_n = 2^n + 1 \]
for all $n = 1, 2, 3, \cdots$.

Proof. Firstly the judgment is clearly true when $n = 1, 2$.

Now assume that, for some natural number $k > 2$, the judgment is true when $n = k - 1$ and when $n = k$. According to the assumption
\[ a_{k-1} = 2^{k-1} + 1, \quad a_k = 2^k + 1. \]

Then
\[ a_{k+1} = 3a_k - 2a_{k-1} = 3 \cdot (2^k + 1) - 2 \cdot 2^{k-1} + 1 = 2^{k+1} + 1 \]
showing that the judgment is true when $n = k + 1$.

It follows from the method of mathematical induction that the judgment is true for all $n = 1, 2, 3, \cdots$.

The fourth variant can be used in the proof of

Example 6.6.12. Let $a$ be a positive number. Then
\[ \frac{1}{a^n} + \frac{1}{a^{n-2}} + \frac{1}{a^{n-4}} + \cdots + a^{n-4} + a^{n-2} + a^n \geq n + 1 \]
for all $n = 1, 2, 3, \cdots$.

Proof. When $n = 1$, the left side is
\[ \frac{1}{a} + a = \left( \frac{1}{\sqrt{a}} \right)^2 - 2 + \left( \sqrt{a} \right)^2 + 2 = \left( \frac{1}{\sqrt{a}} - \sqrt{a} \right)^2 + 2 \geq 2, \]
so that the judgment is true when $n = 1$. When $n = 2$, the left side is
\[ \frac{1}{a^2} + 1 + a^2 = \frac{1}{a^2} - 2 + a^2 + 3 = \left( \frac{1}{a} - a \right)^2 + 3 \geq 3, \]
so that the judgment is true when $n = 2$.

Now assume that the judgment is true for some $n = k - 1$. This means
\[ \frac{1}{a^{k-1}} + \frac{1}{a^{k-3}} + \cdots + a^{k-3} + a^{k-1} \geq k. \]

When $n = k + 1$, the left side of the judgment is
\[ \frac{1}{a^{k+1}} + \frac{1}{a^{k-1}} + \frac{1}{a^{k-3}} + \cdots + a^{k-3} + a^{k-1} + a^{k+1} \geq \frac{1}{a^{k+1}} + k + a^{k+1}. \]

Since the right side of the preceding inequality is
\[ \frac{1}{a^{k+1}} - 2 + a^{k+1} + k + 2 = \left( a^{-(k+1)/2} - a^{-(k+1)/2} \right)^2 + k + 2 \geq k + 2, \]
so that the judgment is true when $n = k + 1$.

It follows from the method of mathematical induction that the judgment is true for all $n = 1, 2, 3, \cdots$.
6.7 Numbers in Peano Structure

Towards the end of the nineteenth century there appeared two works in the domain of foundation of mathematics. The one is Frege’s *Begriffsschrift* published in 1879 and the other is Giuseppe Peano’s *The Principle of Arithmetic, Presented by a New Method* written in Latin and published 1889 (for English translation see Heijenoort, 1967). Both Frege and Peano had the same goal. The goal is to place Arithmetic on a solid logic foundation by inventing a formalized language for logic. Valuation on their achievement, in particular, whether they had achieved the goal, is not my concern here, but in the hands of philosophers of mathematics. As to the languages invented, Peano was the lucky one, for his symbolic language was accepted by a majority of mathematical community, while Frege’s was never taken seriously except in some historical context.

Peano’s original system consists of nine axioms in (Heijenoort, 1967, p. 94), but according to the modern rigor only Axioms 1, 6, 7, 8, 9 can be treated as genuine axioms in Arithmetic. These five axioms together constitute Peano’s system of axioms for Arithmetic, or in another word, Peano structure of natural numbers. For the sake of further discussion let us write down the five axioms. To this end let $\mathbb{N}$, as above, be the set of all natural numbers (for Peano 0 is not a natural number). Thus Peano’s system of axioms is

\begin{itemize}
  \item (P1) $1 \in \mathbb{N}$,
  \item (P2) for each $n \in \mathbb{N}$ there exists a $n' \in \mathbb{N}$ called successor of $n$,
  \item (P3) for each pair $m,n \in \mathbb{N}$ it holds that $m = n$ if and only if $m' = n'$,
  \item (P4) for each $n \in \mathbb{N}$ it holds $n' \neq 1$,
  \item (P5) $E \subset \mathbb{N}$ containing 1 and $n'$ whenever $n \in E$ must be whole $\mathbb{N}$.
\end{itemize}

The last axiom (P5) is also called the principle of (mathematical) induction. In most of modern presentations of the system, the uniqueness of successor is moved from (P3) to (P2). A discussion on the legitimacy of the Peano structure as the foundation of mathematics is philosophical, but no matter what comes out of the discussion, one thing is certain, the Peano structure is a synthesis of intuitive numbers and formalized numbers.

In mathematics as well as in mathematics education, from the Peano structure to that part of mathematics we are experiencing daily, a gap needs to be filled. This was done with grace by Edmund Landau in the book *Grundlagen der Analysis* published in 1930 (for English translation see Landau, 1951). The gap in Landau’s mind is a collection of 301 theorems. The whole collection is mathematization of things we do and use, things we teach and learn, things that we think we know and yet can not
explain adequately. Such mathematization is valuable for modern mathematics education and mathematics teacher education. It is healthy for any serious student in mathematics teacher education to have some knowledge of this little book, in which a logical line, between what are given and what can be deduced, is drawn. Without such a line, a danger of falling into a mathematical vicious circle is overwhelming. Thereby I shall offer here a glimpse of the structure of Landau’s book.

The first lines of the book is: “We assume the following to be given: A set (i.e. totality) of objects called natural numbers, possessing the properties—called axioms—to be listed below.”. What was listed was exactly the Peano structure, except for moving the uniqueness of successor from (P3) to (P2). Before Landau started the whole enterprise of filling the gap, he axiomatized the sign =. Three additional properties of the sign, =, are taken as granted:

- (E1) for all \( n \in \mathbb{N} \) it holds \( n = n \),
- (E2) if \( n = m \) then \( m = n \),
- (E3) if \( n = l \) and if \( l = m \) then \( n = m \).

In order to use the principle of contradiction properly; the negation of tan equality \( x = y \) is assigned to be the inequality \( x \neq y \) and vice versa.

With (P1) – (P5), (E1) – (E3) and the principle of contradiction in hands, the enterprise can begin. The first theorem and the second theorem in the book are

**Theorem.** If \( m \neq n \) then \( m' \neq n' \).

**Proof.** The proof happens to be very simple. Indeed, from \( m' = n' \) it would follow from (P3) that \( m = n \) which contradicts to \( m \neq n \) which is assumed. Therefore \( m' = n' \) is in falsity then its negation \( m' \neq n' \) is in truth.

**Theorem.** Let \( n \) be a natural number. Then \( n \neq n' \).

**Proof.** There seems to exist no other proof of this very simple result than the one based on the principle of induction (P5). Let us look at the content of the theorem and do some analysis first. The first sentence is clearly not any judgment, function of it is simply to explain the meaning of the symbol \( n \), that is the symbol \( n \) represents a natural number, say 89. In mathematical context and by the principle of contradiction, a mathematical object is either in truth or else its negation is in truth. In our case the given object is \( n \neq n' \). Anyone can claim that either \( n \neq n' \) is in truth or else \( n = n' \) is in truth, for the latter is the negation of the former. So, in front of us, we have two mathematical objects

\[ n \neq n', \quad n = n', \]
and we have to judge which one is in truth. Since \( n \) is an arbitrary natural number, we have actually a sequence of two objects
\[
1 \neq 1', \quad 1 = 1', \\
2 \neq 2', \quad 2 = 2', \\
\cdots ,
\]
we have to judge in each line which of the two objects is in truth. If we judge by human reason, then the best we can do is this. We can claim that part of objects in the first column are in truth or else all of objects in the first column are in truth. Now we have a thesis and an antithesis. Notice that the assertion of all of objects in the first column are in truth is equivalent to the theorem to be proved. The concept of set therefore unites this pair of thesis and antithesis.

Let \( T \) be the set (the set of truth) of all natural number \( n \) to which \( n \neq n' \) is in truth, in symbols,
\[
T = \{ n : \ n \in \mathbb{N}, \ n \neq n' \}.
\]
Our theorem is consequently equivalent to the statement that \( T = \mathbb{N} \). This is a typical turn of Theory of Sets, to prove a theorem becomes to estimate the size of a set. In particular, if we want to prove a given set of natural numbers is the full set \( \mathbb{N} \), then we can try to do it by appealing to the principle of induction (P5). This is done in two steps.

By (P1), \( 1 \in \mathbb{N} \), and then by (P4), \( 1' \neq 1 \), so that \( 1 \in T \), this is the first step. In particular, \( 1 \in T \) implies that \( T \) is not empty. Now we turn to the second step. Let us pick an arbitrary natural number \( k \) in \( T \). Being in \( T \) means that \( k \neq k' \) which gives \( k' \neq (k')' \) by the theorem just proven. It follows of (P2) that \( k' \in \mathbb{N} \) which, together with \( k' \neq (k')' \), shows that \( k' \in T \). Now it is the turn of the principle of induction (P5) which says of \( T = \mathbb{N} \), thereby the proof is complete.

The proof given above is a proof by the method of mathematical induction, but it performs from the perspective of Theory of Sets. The new way of performing the induction consists not in any improvement on technical part but in transforming a weak assertion of \( P_k \) is true for some number \( k \) into a strong assertion of pick a number \( k \) in \( T \). This transformation removes nearly all possible psychological obstacles caused by transcending to the infinity.

As I mentioned earlier there are 301 theorems in (Landau, 1951), among which 26 theorems are proved by the method of mathematical induction. The numbers that enumerate these theorems are 2-9, 27-31, 113, 274-5, 279, 281-3, 287-9, 292, 297 and 299.

An essential task in (Landau, 1951) is to change the underpinnings upon which the usual addition is built from counting to the Peano structure. This is done in the fourth theorem of that book.
Theorem. To every pair of natural numbers \( m, n \), we may assign in exactly one way a natural number, called \( m + n \), such that

1. for each \( n \) it holds \( n + 1 = n' \),

2. for each \( m \) and each \( n \) it holds \( m + n' = (m + n)' \).

The proof is made of a double use of the method of mathematical induction and shall not be produced here. And then the additive associativity and the additive commutativity are proved in Theorem 5 and Theorem 6 respectively. Theorem 9 is preparing for the subtraction of type say \( 7 - 5 \). In order to complete the subtraction, an order relation is introduced. For two natural numbers \( m, n \) we say that \( n \) is greater than \( m \), in symbols, \( n > m \), if \( n = m + k \) where \( k \) is another natural number. Theorem 10 is a trichotomy of \( m, n \), that is to say, of

\[ m = n, \quad m < n, \quad m > n \]

only one is in truth in which case the other two are in falsity. Theorem 11 to Theorem 26 concern the usual properties of inequalities, among which, Theorem 12 says that if \( m < k \) and \( k < n \) then \( m < n \), Theorem 19 states that if \( m < n \) or \( m = n \) then \( m + k < n + k \) or \( m + k = n + k \) respectively.

Then comes a very important Theorem 27:

Theorem. In every non-empty set of natural numbers there is a least one.

This is of course that we call the minimum structure of natural numbers. Theorem 28 is about multiplication, its statement is similar to that of Theorem 4 and its proof is *mutatis mutandis* of that of Theorem 4. Theorem 29 and Theorem 31 are about the multiplicative commutativity and the multiplicative associativity respectively. Theorem 30 is about the distributivity, and Theorem 32-36 concern some standard behavior of inequalities under the multiplication. All these constitute the first chapter titled by *natural numbers*. The knowledge obtained in it is a backbone of mathematics.

The piece of mathematics presented in Landau’s book is extremely rare, and hardly can be found in any other textbooks. The material in the rest of the book, on the other hand, is quite standard, and can be found, not so coherent as in Landau’s book, in many modern textbooks.

The second chapter is a construction of positive rational numbers, this construction usually appears in a construction of quotient field from an integral domain. The third chapter is a theory on *cuts*, a turn into the Theory of Sets. The theory of cuts was invented by Dedekind in nineteenth century.
right, one of the sets is the so-called cut. However, it is popular to use the left open one as the cut of a number. With the theory of cuts, all positive real numbers are constructed in the third chapter in (Landau, 1951). That special number zero and all negative real numbers are introduced axiomatically at the beginning of the fourth chapter, and from that moment on, real numbers make an entrance into the stage of mathematics. The arithmetic and the ordering are transplanted in the field of real numbers without much effort, and finally the completeness of real numbers is proved in Theorem 205. The last chapter (the fifth) is about complex numbers. It is interesting to note that existence of irrational numbers is showed in Theorem 162, and this fact is usually proved in text-books by proving that $\sqrt{2}$ is irrational.

Landau’s book is definitely a classic in the literature of mathematical world. However, a classic is usually a book known to everyone but read by no one. The book is a fine collection of didactic transpositions, but those transpositions are not created in the average didactic relation. In fact they are the results of an extreme didactic relation consisting of a gifted mathematician, a gifted student, numbers in Peano, and in addition, the relation is completely free from any institutional duties.

I have seen many collections of didactic transpositions resulting from other extreme. They are sold well and used widely. But this does not mean Landau’s book has less value, for it can not be compared with others. In order to compare two collections of didactic transpositions, they must be created in the same didactic relation.

Any teacher who wishes to do some didactic transpositions from whatever didactic relation must be aware of extremes. The same philosophy is also held by Brousseau (1997) on didactical situations, one must be aware of the Topaze effect and the Jourdain effect. It is only through knowing the extremes we become aware of what we should do and what we should avoid in a given didactic relation. They are conditions and constraints. It is only through knowing the extremes we can avoid a totalization which is self-destructive.

Landau’s book is a treasure in the literature of mathematical world. It is fundamental for it contains all the necessary blocks above the system of Peano to make a mathematics edifice safe. The book is at the same time a valuable collection of fine didactic transpositions based on an extreme didactic relation.

Local issues such as natural numbers, negative numbers, fractions, algebra and geometry are important in mathematics education; interrelationships among such issues are important as well. Ignorance of these interrelationships can only lead to artificial and sometimes unnatural arrangements of local issues. A good principle of searching such interrelationships is to arrange local issues in a dialectical fashion. Dialectical interrelationships are in fact the most natural relationships among mathematical ideas and structures, ans at the level of education they are very useful, sometimes even
vital. I have shown in this chapter such relations for instance among natural numbers, negative numbers and fractions. Dialectical interrelationships are a part of didactic transposition.
Chapter 7

Probability and Statistics

Teaching statistics is a difficult enterprise, especially for a mathematics teacher. At the department where I work, none except for one is willing to teach statistics. A common arrangement is that the course on Probability and Statistics is taught by two teachers.

Any course on Statistics consists in a large collection of empirical methods, ready-made algorithms and theoretical formulas. Levels of deductions on them range from elementary to very high. Terminology used in textbooks is not uniform. All these cry for a didactic order.

The structural analysis offers a framework of organizing probability and statistics education. The main instrument in the organization is the concept of random variable. A discussion on the role random variables play in Probability is conducted in the first part of this chapter. The second part is concentrated on three special issues in Statistics, and a uniform didactic organization is illustrated by examples.

7.1 Probability in Itself

Using the framework of sets Andrei Kolmogorov laid the foundations for a mathematical theory of probability in 1933; the work was published in German and was later translated into English (Kolmogorov, 1950). A fundamental doctrine of the theory of probability held by Kolmogorov is in his own words

The theory of probability, as a mathematical discipline, can and should be developed from axioms in exactly the same way as Geometry and Algebra. This means that after we have defined the elements to be studied and their basic relations, and have stated the axioms by which these relations are to be governed, all further exposition must be based exclusively on these axioms, independent of the usual concrete meaning of these elements and their relations.

(Kolmogorov, 1950, p. 1)
Hence the mathematical theory of probability developed by Kolmogorov is a structural analysis in the sense of this essay. This theory is the universally accepted theory for Probability, therefore the (mathematical) theory of probability in modern literature means exclusively the theory initiated by Kolmogorov.

This theory consists in three functionally different parts \((\Omega, \mathcal{F}, P)\) called probability model or probability triple. The first part is a fixed set \(\Omega\), the second part is a collection \(\mathcal{F}\) of subsets in \(\Omega\), and the third part is a non-negative real function \(P\) defined on \(\mathcal{F}\) with \(P(\Omega) = 1\). The set \(\Omega\) is the universe, an element in \(\mathcal{F}\) an event, and \(P(A)\) the probability of the event \(A\).

As elementary theory of probability, the collection \(\mathcal{F}\) is assumed to be an algebra, and the function \(P\) to be additive. As general theory, the collection \(\mathcal{F}\) is assumed to be a \(\sigma\)-algebra, and the function \(P\) to be \(\sigma\)-additive.

A collection \(\mathcal{F}\) is an algebra if \(A \cup B \in \mathcal{F}\) and \(A \setminus B \in \mathcal{F}\) whenever \(A, B \in \mathcal{F}\) and if \(\Omega \in \mathcal{F}\). A collection \(\mathcal{F}\) is \(\sigma\)-algebra if it is an algebra and in addition if \(A_1 \cup A_2 \cup \cdots \in \mathcal{F}\) whenever \(A_1, A_2, \cdots \in \mathcal{F}\).

A function \(P\) defined on \(\mathcal{F}\) is (finitely) additive if

\[
P(A \cup B) = P(A) + P(B)
\]

whenever \(A\) and \(B\) are in \(\mathcal{F}\) and are disjoint. It is \(\sigma\)-additive if \(P(A) = P(A_1) + P(A_2) + \cdots\) whenever \(A = A_1 \cup A_2 \cup \cdots\) and the union is disjoint.

The theory formulated so far utilizes only one of the four operations in Arithmetic and therefore is a part of Measure Theory. This theory becomes the theory of probability only after the other three operations shall have been introduced.

The subtraction enters into the theory of probability via the so-called complement theorem stating

\[
P(A^*) = 1 - P(A).
\]

It is so because of \(A^* = \Omega \setminus A\) and \(P(\Omega) = 1\).

The multiplication enters into the theory of probability via the concept of independence. Two events \(A\) and \(B\) are called independent if

\[
P(A \cap B) = P(A)P(B).
\]

The division enters into the theory of probability via the concept of conditional probability and the so-called total probability theorem. Let \(A\) and \(B\) be two events with \(P(B) > 0\). Then the number

\[
P(A|B) = \frac{P(A \cap B)}{P(B)}.
\]
is called conditional probability of $A$ given $B$. The total probability theorem states

$$\Pr(E) = \Pr(E|A_1)\Pr(A_1) + \Pr(E|A_2)\Pr(A_2) + \cdots + \Pr(E|A_n)\Pr(A_n)$$

if $A_1 \cup A_2 \cup \cdots \cup A_n = \Omega$ is a disjoint union.

What we have discussed is the very foundation of the mathematical theory of probability, although the theory has been developed by leaps and bounds, this foundation remains nearly the same. This can be seen clearly from Kolmogorov’s book (1950) and Blom’s book (1989).

## 7.2 Probability for Others

In Arithmetic, a number in itself or a number as a noumenon is a pure number, and as a result the emphasis is not so much on its ontological meaning but on its operational relations to other numbers. Such relations, as mathematical theory, are postulated as axioms through the four operations. Thus any two numbers can be added together and any number can be multiplied by any other number to yield another number. These two operations are subjected to the three laws. The inverse of them leads to the subtraction and division.

In arithmetic education, a number for others or a number as a phenomenon is a number embodied in others. Thus after the number 5 has been embodied into book, it becomes five books. As a phenomenon, the number 5 not only can be embodied into book but also any other material thing say apple. There are two ways to express the epistemological relationship among the number 5, the five books and the five apples.

The first way is the stable and commutative diagram:

```
<table>
<thead>
<tr>
<th>5</th>
<th>II</th>
<th>five apples</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td></td>
<td>five books</td>
</tr>
<tr>
<td>III</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

The second way is the sense of invariance in the process in which the set of five books and the set of five apples struggle to pass into each other. This sense is the sense of being equal in quantity, and the quantity (amount) of the set of five books is thus the number 5. The principal instrument of acquiring such sense is the arithmetic intuition.

In Geometry, a point in itself or a point as a noumenon is a pure point, and as Euclid did, the emphasis is not so much on its ontological meaning but on its operational relations to others. Thus; a point is that which has no part, and a line is breadthless length, and a surface is that which has length...
and breadth only. Relations among points, lines and surfaces, as geometrical theory, are postulates. For instance, to draw a straight line from any point to any point.

In geometry education, a point for others or a point as a phenomenon is a point embodied in position. Thus a point can be here or there, a triangle can be drawn on this piece of paper or on that piece of paper. The sense of invariance in the process of moving any figure from here to there is the sense of being equal in position. This sense is also the underpinnings in the Klein’s Erlangen program. As in the case of Arithmetic, this can be expressed symbolically as

\[
\begin{array}{c}
\text{figure} \\
\text{I} \\
\text{III}
\end{array} \xrightarrow{\text{II}} \begin{array}{c}
\text{there} \\
\text{II} \\
\text{III}
\end{array} \xrightarrow{\text{here}} \begin{array}{c}
\text{here}
\end{array}
\]

The principal instrument of acquiring such sense is the geometric intuition.

In Probability, probability in itself or probability as a noumenon is a probability model introduced above.

In probability education, a probability model as a phenomenon manifests in different empirical models. Just like in Arithmetic and Geometry, interactions among different empirical models lead to certain sense of invariance. This sense is the sense of being equal in probability. The principal instrument of acquiring such sense is the probabilistic intuition. The sense of being equal in probability, in terms of random variables, shall be discussed in what follows.

### 7.3 Probability for Itself

The synthesis of probability in itself and probability for others is probability for itself. The mathematical instrument leading to the synthesis is the concept of random variable or what amounts to the same as stochastic variable. The concept is uncomplicated, but it takes some time and energy to become familiar with it (Blom, 1989, p. 40). It is uncomplicated only in the mathematical theory of probability, for a random variable is a measurable function defined on the set \( \Omega \). In probability education, this definition of random variable is seldom taken as the dominating one. Instead, some heuristic motivation for such a variable is often taken as the definition. In the case of Blom for instance, such a heuristic definition runs as follows.

A random trial often produces a number associated with the outcome of the trial. This number is not known beforehand; it is determined by the outcome of the trial, that is, by chance. It is therefore called a random variable (1989, p. 40).
From the operational perspective, the concept of random variable corresponds to that of probabilistic model. The certainty in such a model is the set $\Omega$ and it corresponds to the concept of being random variable $X$. Thus $P(\Omega) = 1$ becomes $P(-\infty < X < +\infty) = 1$. The knowledge about $F$ is not needed completely, instead, a special type of interval is in focus, the type of interval is $(a, b]$. The probability $P(A)$ where $A \in F$ is replaced by

$$P(a < X \leq b).$$

Let us express such a scheme symbolically as follows:

\[
\begin{align*}
\Omega & \quad \longleftrightarrow \quad \text{being } X \\
\mathcal{F} \ni A & \quad \longleftrightarrow \quad (a, b] \subset \mathbb{R} \\
P & \quad \longleftrightarrow \quad P.
\end{align*}
\]

In tabulation the probability $P(a < X \leq b)$ is not suitable, for it depends on two numbers $a$ and $b$. This restriction can be removed by introducing

$$F_X(x) = P(X \leq b)$$

and call it (probability) distribution of $X$. The relationship between a random variable and its distribution is written as

$$X \in F_X(x).$$

Of course, it holds that

$$P(a < X \leq b) = F_X(b) - F_X(a).$$

It is safe to say that Probability and Statistics is the study of random variables and their distributions.

Random variables have principally two kinds — continuous and discrete. In the case of continuous random variable, the derivative of a distribution is sometimes more suitable than distribution itself. The derivative

$$f_X(x) = F_X'(x)$$

is called density function of the random variable $X$. With the density function we have

$$P(a < X \leq b) = \int_a^b f_X(x) \, dx.$$ 

There is no better way to express all these than the following figure, in which the probability that $a < X \leq b$ is the area of the shadowed domain:
which is absolutely one of the most important didactic transposition of random variable. As a special case we express the distribution by the density function in the following fashion

\[ F_X(x) = \int_{-\infty}^{x} f_X(t) \, dt. \]

The discrete case is even simpler for the integral is changed into a sum and the density function is changed into a probability function \( p(x) = P(X = x) \). Thus the preceding didactic transposition becomes

\[ P(a < X \leq b) = p_5 + p_6 + p_7 \]

Of all random variables two are singularly important. The one is the continuous variable with the density function:

\[ y = \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \]

and such a variable is denoted by \( X \in N(0, 1) \). The other is the discrete variable with the probability function:

\[ P(X = k) = \binom{n}{k} p^k q^{n-k} \]

and such a variable is denoted by \( X \in \text{Bin}(n, p) \).

Using the framework constructed so far, the main issues in probability and statistics education can now be formulated. The fundamental objects in Probability and Statistics are random variables taken as structures, and therefore the method in it is the structural analysis. That is Probability and Statistics concerns itself interrelationships among random variables. These relationships are expressed in terms of distributions of random variables. Two distributions are particularly significant, and these are the normal distribution and the binomial distribution, and the relationships to them are in focus.
Interrelationships among random variables other than logical ones are especially important in probability and statistics education, for it is here that the sense of being equal in probability shall play a major role, and it is here that the probabilistic intuition shall be trained and developed.

In Probability, a random variable as a noumenon, is exactly like a number in Arithmetic or a point in Geometry, can be observed via phenomena. It is therefore very important to develop a sense of being equal in probability, or what amounts to the same as to judge whether two observed phenomena has the same noumenon. Such a judgment is not rooted in the logical reason but in the probabilistic intuition.

The length and the weight of a randomly chosen person are random variables. The exact weight of a randomly chosen package of butter marked 500 g is another random variable. The life-time of a randomly chosen electric bulb is a random variable. However, the first three random variables have the normal distribution but the last one does not have. This can only be judged by the probabilistic intuition.

Assume that the student takes a certain examination nine times and each time the student has 30% chance to pass the examination. Then the number of passes in these nine tries is a random variable. A urn contains three white balls and seven black balls. A person draws nine balls with replacement. Then the number of white balls among these nine balls drawn is a random variable. It is the probabilistic intuition that judges that these two random variables are the same binomial variable, therefore they are equal in probability.

The probabilistic intuition, expressed as the sense of being equal in probability, is very different from the arithmetic intuition, expressed as the sense of being equal in quantity, and from the geometric intuition, expressed as the sense of being equal in position. The former concerns itself a compound of many parts with the emphasis on their proportional relation, while the latter concern themselves only some simple relation among point-like objects. The concept of random variable is the most important instrument in acquiring such intuition, for random variables are allowed to be compared.

7.4 Statistics

In no other place a didactic order is more needed than in Statistics. Here by Statistics I mean Mathematical Statistics at beginning level in any university. The students to whom Statistics is taught can be those on mathematics, or on engineering or on economics. Didactic issues in statistics education are simply not on the agenda of mathematics education researchers. For instance, in (Freudenthal, 1974) and the chapter 18 of (Freudenthal, 1973) no discussion is made on Statistics, though Probability and Statistics is the title of the chapter. Therefore no attempt is made in this chapter to conduct
a didactic discussion on Statistics as a whole. Instead, some discussions on some issues in Statistics shall be made within the framework of the present essay. And I choose three special issues – the point estimation, the interval estimation and the testing hypothesis. However, before that, I shall offer here a discussion on the possible reason that distinguishes statistics education from others.

In some sense Statistics is the inverse of Probability. Inverses in mathematics prevail everywhere, thus, the square root is the inverse of square, the logarithm is the inverse of exponential, the integral is the inverse of derivative. These examples show that the inverse of a thing transcends that thing. This means, in the case of square, the construction of square root is beyond the multiplication which constructs the square. Therefore knowledge on Statistics is constructed upon knowledge in Probability as a whole and knowledge in other areas, especially in Analysis. One of the major tasks in Statistics is to make decision based on empirical data, and therefore, it contains variety of empirical algorithms and methods, such as the method of maximum likelihood, the method of least squares, and the $\chi^2$ method. In order to carry out the first two methods, the whole differential calculus is needed; while although the $\chi^2$ method is easy to use, but it is didactically impossible to deduce the method rigorously at undergraduate level.

Therefore, from didactic point of view, it is important to classify knowledge into two classes. The one class is the knowledge at undergraduate level which the student should master, and the other class is the knowledge at the higher level which the student should be able to make use of it but postpone true understanding to a later occasion. The student should be aware of the classification with help of the teacher. At the place where a true understanding has to be postponed, some characteristic examples are needed to serve an intermediate stage.

Based on the discussion made above, I believe the following points are didactically indispensable in statistics education at the beginning level of a university:

- schemata
- characteristic examples
- construction upon the characteristic examples
- differentiation between the present level and later level

I shall later show the use of these points in teaching on point estimation, on interval estimation and on testing hypothesis, but for the preparation, I shall first discuss an important concept of independent random variables of identical distribution, or in short, i.i.d.,
7.5 Independent Identical Distributions

Technically, the statistics concerns itself certain unknown parameters $\theta$ in a random variable $X$, for instance, $\theta$ is the expectation or standard deviation of $X$. Let us assume $X$ is the length variable in certain country. In order to estimate $\theta$, we have to observe the random variable $X$. In practice, this means that the length of some randomly chosen persons in that country are measured. The results are

$$x_1, x_2, \cdots, x_n.$$

A dilemma arises immediately. The $n$ persons can not be measured at exactly same time on the one side, and a genuine statistics must measure $n$ persons at the same time on the other side. The same dilemma appears in statistical mechanics as well, and the dilemma was solved by Gibbs. He introduced a theoretical apparatus, the independently mental copies

$$X_1, X_2, \cdots, X_n$$

of $X$. Therefore all $X_k$ have the same distribution of $X$. Instead of observing $X$ at $n$ different times, now we observe $X_1, X_2, \cdots, X_n$ at the same time. The famous ergodic hypothesis states the statistical results based on these two kinds of sampling are the same. The above mental copies $X_1, X_2, \cdots, X_n$ are called independent random variables of identical distribution, in short, they are i. i. d. random variables.

7.6 Point Estimation

7.6.1 Schema

Let $X$ be a random variable and $X_1, X_2, \cdots, X_n$ be independent copies of $X$. The aim of the point estimation is to estimate the unknown parameter $\theta$ in $X$ by finding another random variable $Y$, called the sample variable. Of course $Y$ should be a function of $X_1, X_2, \cdots, X_n$, therefore we write

$$Y = \theta^*(X_1, X_2, \cdots, X_n) \quad \text{or simply} \quad Y = \theta^*(X).$$

A point estimate of $\theta$ is an observation

$$\theta^*(x_1, x_2, \cdots, x_n) \quad \text{or simply} \quad \theta^*(x).$$

of the sample variable. We represent it by the following picture:

$$\theta \xrightarrow{\theta^*(x)} \theta$$
The important concepts of unbiasedness, consistency and efficiency are easy to define but difficult to verify, for they depend on among others the law of large numbers.

Two first principles are used in finding sample variables. The one is the method of maximum likelihood in which a sample variable is found by maximizing the likelihood function $L(\theta)$. The other is the method of least squares in which a sample variable is found by minimizing the function $Q(\theta)$.

Let $f(x; \theta)$ be the density function of $X$ in the continuous case, and $p(x; \theta)$ be the probability function of $X$ in the discrete case, and let $m(\theta)$ be the expectation of $X$. Then

$$L(\theta) = \begin{cases} f(x_1; \theta) \cdot f(x_2; \theta) \cdot \cdots \cdot f(x_n; \theta) \\ p(x_1; \theta) \cdot p(x_2; \theta) \cdot \cdots \cdot p(x_n; \theta). \end{cases}$$

and

$$Q(\theta) = (x_1 - m(\theta))^2 + (x_2 - m(\theta))^2 + \cdots + (x_n - m(\theta))^2.$$ 

### 7.6.2 Characteristic Examples

The simplest distribution is the two-point distribution, and a random variable with two values has such distribution. Thus let $X$ be a random variable with values $a$ and $b$, and let $p = P(X = a)$ and $q = P(X = b)$ be its probability function. Clearly $p + q = 1$. We assume the unknown parameter $\theta$ is $p$.

To find a sample variable $p^*(X)$ by the method of maximum likelihood, we proceed as follows. Copy $X$ into $n$ independent copies $X_1, X_2, \cdots, X_n$ and observe them. The result is $x_1, x_2, \cdots, x_n$ of which we assume that $k$ are $a$ in which case the rest is $b$. Then the likelihood function is

$$L(p) = p(x_1) \cdot p(x_2) \cdot \cdots \cdot p(x_n) = p^k (1 - p)^{n-k}.$$ 

Since $p = P(X = a)$ then $p$ varies over the interval $[0, 1]$. The differential calculus leads quickly to the result that the maximum of $L(p)$ where $p \in [0, 1]$ is attained at $p = k/n$. Hence the sample variable is

$$p^*(X) = p^*(X_1, X_2, \cdots, X_n) = \frac{X_1 + X_2 + \cdots + X_n - nb}{n(a - b)} = \tilde{X} - b.$$

To find a sample variable $p^*(X)$ by the method of least squares, copy $X$ into $n$ independent copies $X_1, X_2, \cdots, X_n$ and observe them. The result is $x_1, x_2, \cdots, x_n$ of which we assume that $k$ are $a$ in which case the rest is $b$. Then

$$Q(p) = k(a - b - p(a - b))^2 + (n - k)(p(a - b))^2$$

for $m(p) = E(X) = ap + b(1 - p) = b + p(a - b)$. Since

$$Q(p) = (a - b)^2 (k - 2kp + np^2)$$
and since $0 \leq p \leq 1$, it follows from the standard technique in differential calculus that the minimum of $Q(p)$ is attained at $p = k/n$. This result coincides with that obtained above, and thereby the sample variable obtained by the method of least squares is the same as that obtained by the method of maximum likelihood.

## 7.7 Interval Estimation

### 7.7.1 Schema

Let $\theta$ be as before the unknown parameter in a random variable $X$. The aim of the interval estimation is to cover the unknown parameter $\theta$ in $X$ by a confidence interval. This means we have to find two random variables $Y$ and $Z$, called confidence limits, so that

$$P(Y < \theta < Z) = 1 - \alpha.$$ 

Of course $Y$ and $Z$ should be functions of $X$, so we write $a_1(X) = Y$ and $a_2(X) = Z$. The above formula becomes

$$P(a_1(X) < \theta < a_2(X)) = 1 - \alpha.$$ 

After observation, the interval $I_0 = (a_1(x), a_2(x))$ is called confidence interval for $\theta$ with confidence level $1 - \alpha$.

The education on interval estimation at beginning level of a university should concentrate on $m$ or $\sigma$ in $X \in N(m, \sigma)$.

### 7.7.2 Characteristic Examples

We shall find a confidence interval for $m$ in $X \in N(m, \sigma)$ under the assumption that $\sigma$ is known, and this shall be the characteristic example. Let $X_1, X_2, \ldots, X_n$ be independent copies of $X$. Since $m$ is the average, it leads us to consider a random variable $\bar{X}$. A theorem in Probability states that

$$T = \frac{\bar{X} - m}{D} \in N(0, 1), \quad D = \frac{\sigma}{\sqrt{n}}$$

and the density function of it is $\varphi(x)$. Consider the figure:
and hence
\[ P(-\lambda_{\alpha/2} < T < \lambda_{\alpha/2}) = 1 - \alpha. \]

It follows that
\[ P(\bar{X} - D \cdot \lambda_{\alpha/2} < m < \bar{X} + D \cdot \lambda_{\alpha/2}) = 1 - \alpha. \]

The confidence limits are
\[ a_1(X) = \bar{X} - D \cdot \lambda_{\alpha/2}, \quad a_2(X) = \bar{X} + D \cdot \lambda_{\alpha/2}, \]

and
\[ I_m = (\bar{x} - D \cdot \lambda_{\alpha/2}, \bar{x} + D \cdot \lambda_{\alpha/2}) \]
is a confidence interval for \( m \) with confidence level \( 1 - \alpha \).

Now we construct \( I_m \) under the assumption that \( \sigma \) is unknown, and the construction is modeled on the characteristic example.

Since \( \sigma \) is unknown, we replace it by \( s \). Since
\[ s = \sqrt{\frac{\sum_{k=1}^{n}(x_k - \bar{x})^2}{n - 1}}, \]
we replace the random variable \( T \) in the characteristic example by
\[ T = \frac{\sqrt{n}(\bar{X} - m)}{\sqrt{\sum_{k=1}^{n}(X_k - \bar{X})^2}}. \]

It follows from a theorem in Probability that the density function of the preceding \( T \) is
\[ f_T(x) = \frac{\Gamma(n/2)}{\sqrt{\pi(n-1)\Gamma((n-1)/2)}} \cdot \left(1 + \frac{x^2}{n-1}\right)^{-n/2}. \]

We write it as
\[ T \in t(f), \quad f = n - 1. \]

Consider the figure:
and hence
\[ P(-t_{\alpha/2}(f) < T < t_{\alpha/2}(f)) = 1 - \alpha. \]
Setting
\[ a_1(X) = \bar{X} - \sqrt{\sum_{k=1}^{n}(X_k - \bar{X}^2) \over n-1} \cdot t_{\alpha/2}(f) / \sqrt{n} \]
and
\[ a_2(X) = \bar{X} + \sqrt{\sum_{k=1}^{n}(X_k - \bar{X}^2) \over n-1} \cdot t_{\alpha/2}(f) / \sqrt{n} \]
it follows that
\[ P(a_1(X) < m < a_2(X)) = 1 - \alpha. \]
Hence
\[ I_m = (\bar{x} - d \cdot t_{\alpha/2}, \bar{x} + d \cdot t_{\alpha/2}), \quad d = {s \over \sqrt{n}} \]
is a confidence interval for \( m \) with confidence level 1 - \( \alpha \).

7.8 Hypothesis Testing

7.8.1 Schema

Let \( X \) be as above a random variable containing a unknown parameter \( \theta \). Let \( X_1, X_2, \cdots, X_n \) be independent random variables having the same distribution as \( X \) has (independent copies). Insofar as the unknown parameter exists, any numerical calculation is out of question. Hence if we insist in doing some numerical calculations then we must give some values to the unknown parameter. Supplying the unknown parameter with values does not have any scientific evidence, thereby can only be treated as a hypothesis.

Let us take a simple case, that is we assume \( \theta = \theta_0 \). This kind of assumption in statistics has only one purpose, that is to make some computations possible, therefore it has a special name – null hypothesis, often written as

\[ H_0 : \theta = \theta_0. \]

After \( \theta \) gains the value \( \theta_0 \), theoretically, computations can be carried out not only for the random variable \( X \) but also for any function of \( X \). In statistical practices, the computations shall be carried out for some well chosen variable \( t(X) \) called sample variable. The computations are of special kind, the so-called small probability computations. To be precise, a set \( C \), called critical region, is searched in such way that

\[ P(t(X) \in C) \approx \alpha. \]
In this computation the number $\alpha$ is small, often chosen to be 0.05, 0.01 and 0.001. The preceding equation is popular in Statistics, but not very suitable for teaching. The following one is better

$$P(\theta = \theta_0, t(X) \in C) \approx \alpha.$$ 

Let us interpret this formula. Let us first of all make a observation $t(x)$ of $t(X)$ and such a observation is called test quantity. Let us assume that the test quantity falls into the critical region, that is

$$t(x) \in C.$$ 

Under these circumstances the probability that $\theta = \theta_0$ is small $\alpha$, hence should be rejected. This is the whole philosophy behind testing hypothesis. And now we shall formalize it. The name of this kind test is test of significance which says that

- if $t(x) \in C$ then reject $H_0$,
- if $t(x) \notin C$ then do no reject $H_0$.

In addition, we say that the test is done at level $\alpha$ which is also called level of significance.

### 7.8.2 Characteristic Examples

In my hand I have a coin so as $P(H) = p$. I want to test the hypothesis that the coin is fair, that is

$$H_0 : p = 0.5.$$ 

I shall test this hypothesis by throwing it independently 15 times. Let $x$ be the number of heads in these 15 throws. Then $x$ is a observation of $X \in \text{Bin}(15, p)$. Under the assumption $p = 0.5$ computations can be carried out for $t(X) = X$. Since the probability function of $X$ is small near 0 and 15, so that a critical region must be searched there. For instance I shall compute some small probabilities near 0.

We know that $P(X \leq 3) = 0.05923$ and therefore take the critical region as $C = \{0, 1, 2, 3, 4\}$. Hence we design the following test:

- if $x \leq 4$ then reject $H_0$,
- if $x > 4$ then do no reject $H_0$.

This test is at level 0.06.

We know that $P(X \leq 3) = 0.01758$ and therefore take the critical region as $C = \{0, 1, 2, 3\}$. Hence we design the following test:
• if $x \leq 3$ then reject $H_0$,
• if $x > 3$ then do no reject $H_0$.

This test is at level 0.02 or is significant*.

We know that $P(X \leq 1) = 0.00049$ and therefore take the critical region as $C = \{0, 1\}$. Hence we design the following test:

• if $x \leq 1$ then reject $H_0$,
• if $x > 1$ then do no reject $H_0$.

This test is at level 0.0005 or is significant**.

We know that $P(X \geq 13) = 0.00369$ and therefore take the critical region as $C = \{13, 14, 15\}$. Hence we design the following test:

• if $x \geq 13$ then reject $H_0$,
• if $x < 13$ then do no reject $H_0$.

This test is at level 0.00369.

We know that $P(X \leq 3 \cup X \geq 13) = 0.02127$ and therefore take the critical region as $C = \{0, 1, 2, 3, 13, 14, 15\}$. Hence we design the following test:

• if $x \not\in (3, 13)$ then reject $H_0$,
• if $x \in (3, 13)$ then do no reject $H_0$.

This test is at level 0.02.

In our next example the random variable $X$ is the length variable in certain country. We assume that $X \in N(m, 8)$. We want to test the hypothesis

$$H_0 : m = 172.$$  

And we take sample variable as

$$t(X) = \bar{X} = \frac{X_1 + X_2 + \cdots + X_{10}}{10}.$$  

Under the hypothesis $H_0$ we have $\bar{X} \in N(172, 8/\sqrt{10})$ then

$$P\left(172 - 8\lambda_{\alpha/2}/\sqrt{10} < \bar{X} < 172 + 8\lambda_{\alpha/2}/\sqrt{10}\right) = 1 - \alpha$$

and we take the critical region $C$ as the complement of the interval

$$I_\alpha = \left(172 - 8\lambda_{\alpha/2}/\sqrt{10}, 172 + 8\lambda_{\alpha/2}/\sqrt{10}\right).$$

Hence we design the following test:
• if $\bar{x} \notin I_\alpha$ then reject $H_0$,
• if $\bar{x} \in I_\alpha$ then do no reject $H_0$.

This test is at level $\alpha$. If $\alpha = 0.05$ then test becomes

• if $\bar{x} \notin (167.042, 176.958)$ then reject $H_0$,
• if $\bar{x} \in (167.042, 176.958)$ then do no reject $H_0$.

This test is significant*.

This is the last chapter of the present essay, but not the end of the story. In fact, it is a beginning of a didactical route within the framework laid by Brousseau and Chevallard.

If you open a mathematics textbook, you will see a lot of mathematical reasonings, they are expressed by a string of formulas in combination with some natural texts. If you decide to learn something in such a book, then mathematical texts must be in the first place readable. No mathematical text in a book or on a blackboard is the original invention of some mathematicians, instead, a piece of mathematical text is a variant of the original. And the variation is made in order that the original becomes readable. This clearly indicates that some variant is less readable and some more readable. A production of a readable text in a book or on a blackboard is a didactic transposition.

The entire enterprise in this essay is to demonstrate, by a large amount of examples, that didactic transpositions can indeed be generated by two general principles — the syllogistic analysis and the cunning of reason.
Afterword and Acknowledgment

Now, I, as a mathematics researcher and teacher, shall give an end to this essay, which for me is not a string of texts any more. In the first place, the essay is a collection of varieties of experiences of looking at mathematics, from inside and from outside, from teaching and from learning, from individual and from societal. In the second place, it witnesses a process of growing up of a mathematics teacher in the didactical direction. In the third place, it records some typical events of the collection and the process. Thereby, inevitably, some words contained in the essay are emotional, some words are mature and some are childish, some words are said with precision and some lack of that. I myself can not judge them, but at least one thing I can judge, that thing is my honesty. The honesty manifests in helping to promote/improve students’ learning of mathematics and acquisition of mathematical competencies (Niss, 1999). To have such honesty is one thing and to do its justice is another thing, a much harder thing. If there is anything at all that I have learned during the writing of the essay which the honesty calls for, then it is this, to promote/improve students’ learning of mathematics and acquisition of mathematical competencies is a complex of many components which should be taken care of with vigilance. I shall therefore describe in short a way, certainly among others, the students could be helped.

Among many things that a teacher does in a teaching process, the one is the transformation of mathematical knowledge in a textbook into that in the teacher’s own notebook. Knowledge can be transformed in variety of ways. Inasmuch as a teaching process is a social phenomenon, it is subjected to conditions and constraints. The conditions tell the teacher what to do, whereas the constraints tell the teacher what should be avoided. Hence the teacher must take a certain relation of the teacher, the student, the knowledge, and some other institutional duties into consideration. Such a relation is the didactic relation (Chevallard, 1988). In order to fit in this didactic relation, a philosophical positioning of knowledge must be held, knowledge is not that sacred and mysterious any more, knowledge becomes
fragile (Brousseau and Otte, 1991), it changes constantly its form. Each form has a didactical value. Thereby teachability is measured by that value. The higher didactical value the better teachability.

The question of how can a didactical value be raised becomes fundamental. Any answer clearly depends on a didactic relation. In so far as knowledge is concerned, textbooks are the most safe place to preserve and transmit knowledge (Kang and Kilpatrick, 1992). But a textbook is often written in the average didactical relation (pseudo-didactic relation). Such a didactic relation is what a textbook author has in his mind or what a teacher without teaching duty has in his mind. When the teacher tries to raise a didactical value of a piece of knowledge or a body of knowledge, the teacher must do it from a particular didactical relation in which the teacher is placed, thereby any didactic transposition he does depends entirely on that relation. This does not mean the teacher should ignore altogether other didactic transpositions, on the contrary, it is very helpful to take other didactic transpositions based on the average didactical relation into consideration. Didactic transpositions on the average didactical relation can thus become a research field. The works by Freudenthal (1983) and by Stolyar (1974) belong to this field. And I believe the present essay belongs to it as well.

These works immediately suggest a research question of is there other general way of generating didactic transpositions in the average didactic relation. As to the practical issue, I believe no teacher can use the didactic transpositions of Freudenthal directly without any modification, the teacher must combine the principle behind those didactic transpositions with the teacher’s own didactic relation. Thereby, to use didactic transpositions of Freudenthal in practice is to say that it is always helpful to carry out a phenomenological analysis on a piece of knowledge or a body of knowledge to be taught by the teacher. In my case, it is always helpful to carry out a syllogistic analysis combined with the cunning of reason on a piece of knowledge or a body of knowledge to be taught. I know I shall do so, and hope I am not the only one.

I wrote down the first word of the essay in 2008, and since then five years have elapsed. During the time and in the writing, many friends have helped me, in one way or another, consciously or unconsciously. I now utter thanks, a light word with heavy feeling, towards all these friends.

Among all those friends some are particularly significant, and I shall mention them explicitly.

First of all, Docent Arne Engström of Karlstad University and Professor Gunnar Gjone of University of Oslo deserve my special gratitude for being the supervisors of this dissertation. Arne has witnessed the evolution of the essay throughout, and commented on each one of its versions. His constructive criticisms and suggestions have been invaluable to me. In fact it was Arne who introduced me to the French school of didactics. Although I met Gunnar sporadically, his encouragement played an important role in
carrying out this writing project.

It was a fortunate and happy event when I met Professor Xin Bing Luo, then of Shaanxi Normal University, and now of Örebro University and Shaanxi Normal University. The event has led to a fruitful cooperation. Most of time when Xin Bing was in Örebro, our conversation was directed towards the essay; and it has led to improvements of the presentation. To Xin Bing I say in addition 落老许学.

Now I would like mention Universitetslektor Ola Helenius of Örebro University and NCM for being the opponent at the final seminar. His constructive criticism has led to a complete re-writing of the version presented at the seminar. Thanks Ola!

During the writing, I took two courses on Pedagogics and Theory of Curriculum conducted by Professor Tomas Englund and Professor Ninni Wahlström of Örebro University. The courses have opened my eyes towards a territory unknown to me before, for this, I say, thanks Tomas, Ninnie.

I have been being a teacher at the School of Science and Technology at Örebro University since 1995, and the writing became possible only because of the generous support from the leaderships there. In particular, Professor Peter Johansson of Örebro University has been a great help in arranging all administrative formalities. Tack Peter!

My former student and now a PhD student of economics at Uppsala University, Hai Shan Yu, was probably the first audience of this work, and she is also a great help in assisting me with the literature issue. I therefore would like to say to her, 海山学.

To these days I still remember that afternoon when my dear friend Anders Hellgren sat at the institute and read Republic. When I asked why then he explained that that book to certain extent is about didactics. The impact of this event on me is showed clearly in this essay. Anders read also an early version of the text and corrected some of my mistakes on language. Tack Anders!

The writing period happens to coincide with that of early schooling of my children, Elin and Johannes. This supplied me with a golden opportunity in observing continuously and at close range their mathematical thinking developments. Needless to say, countless number of occasions took place among us concerning mathematics teaching and learning. The outcomes have been important to me and are reflected in parts of the text implicitly and explicitly. I defended my first dissertation twenty years ago and soon I shall defend this one. These two dissertations become possible largely because of the existence of my wife, Marie, in my life. Although we do not share what existed in the dissertations, we share the life. My life companied by them is filled with joy. However one thing I shall do now that might disappoint all of you. I shall not write the phrase To ···, you all know I am not a magician. A magician can create such an illusion that a knife seems to be thrown out but in reality that knife never has left his hand for a single second. Instead,
I shall do a real thing which simply can not create any illusion. This real thing is what I do best and mean genuinely in this circumstance, and it is: 

tack, xuè xuè Moj, Yezi, Nanne!
Bibliography


Syllogistic Analysis and Cunning of Reason in Mathematics Education

This essay explores the issue of organizing mathematics education by means of syllogism. Two aspects turn out to be particularly significant. One is the syllogistic analysis while the other is the cunning of reason. Thus the exploration is directed towards gathering evidence of their existence and showing by examples their usefulness within mathematics education.

The syllogistic analysis and the cunning of reason shed also new light on Chevallard’s theory of didactic transposition. According to the latter, each piece of mathematical knowledge used inside school is a didactic transposition of some other knowledge produced outside school, but the theory itself does not indicate any way of transposing, and this empty space can be filled with the former.

A weak prototype of syllogism considered here is Freudenthal’s change of perspective. Some of the major difficulties in mathematics learning are connected with the inability of performing change of perspective. Consequently, to ease the difficulties becomes a significant issue in mathematics teaching. The syllogistic analysis and the cunning of reason developed in this essay are the contributions to the said issue.