Progression of algebraic discourse in school years 2 to 8

Maria Kenneman
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"The power of imagination makes us infinite."

John Muir
Progression of algebraic discourse in school years 2 to 8

Maria Kenneman
Abstract

In contemporary educational settings algebra is considered to be of vital importance for student’s continuation to more advanced studies in mathematics, thereby affecting their chances for future education and employment. However, a substantial number of students do not benefit from the algebra presently taught in schools and fail to use algebraic reasoning. The purpose of this study was to enhance the understanding of how classroom discourse supports the students’ learning of algebra. The study rests on two basic assumptions, firstly mathematics is regarded a discourse, secondly teachers’ instruction during lessons and the textbooks used in school are envisioned as potential means for supporting students’ algebraic development. The issue of learning was examined through a focus on progression of algebraic discourse in mathematics textbooks, for grade levels 2, 5 and 8. Furthermore, in order to study classroom discourse more broadly, the algebraic discourse of teachers’ lesson introduction talks in grade 8, were examined in relation to the algebraic discourse of textbooks. The foundation for the analyses was a discursive perspective and a communicational theory depicting algebraic development as a hierarchical structure of consecutive discursive levels. The mathematics textbooks’ and teachers’ discourses were analysed regarding the presence of signifiers of algebraic objects, more informally called unknowns, and concerning four measures of discursive complexity. Mean value of the number of words constituting the signifier of algebraic object, signifier length equal to or exceeding two words, signifier length equal to or exceeding six words, and as amount of signifiers of algebraic objects of a higher discursive level.

The results show that there were signifiers of algebraic objects present in all three mathematics textbooks and in teachers’ lesson talks. The number of these signifiers of algebraic objects in the mathematics textbooks grew substantially between grade 2 and 5 with a moderate increase between grade 5 and 8. Also the mean value of the number of words constituting these signifiers of algebraic objects grew between grade 2 and 8, as well as the amount of signifiers of algebraic objects consisting of six or more words. Complexity measured as amount of signifiers of algebraic objects of a higher discursive level grew from grad 2, were there were no such signifiers of algebraic objects, to grade 8 were there were 17 % of the total amount. Thus, the analyses of the textbooks exhibit a progression of increasing complexity in terms of the measures focused in this study. In comparison, the complexity of teachers’ discourse is lower than the discourse of any of the mathematics textbooks concerning mean value of signifier length. The amount of signifiers of algebraic objects of a signifier length equal to or exceeding two words were comparable with the amount in the grade 2 mathematics textbook. Concerning signifier length equal to or exceeding six words the amount in the teachers’ lesson talks were in the same order of size as the corresponding measure in the mathematics textbook of grade 5.
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1. INTRODUCTION

1.1. Algebra and discourse

I think I am not alone remembering and associating school algebra with the efforts of mathematical manipulation. Lessons filled with assignments simplifying expressions containing brackets, factoring expressions, and, perhaps most of all, solving equations in tasks manipulating symbols or unknowns. Once you get the hang of it, that is, if you can make do with not knowing the unknown, algebra is a fascinating business. These symbols can make wonders by helping you solve tremendously complicated problems.

*If there is a heaven for school subjects, algebra will never go there. It is the one subject in the curriculum that has kept children from finishing high school, from developing their special interests and from enjoying much of their home study work. It has caused more family rows, more tears, more heartaches, and more sleepless nights than any other school subject.* (Kilpatrick & Izsák, 2008, quoted in Cai and Knuth, 2011, p. vii)

Judging from the quote above, and the vast amount of research studies emphasizing school algebra, not all people get to experience this magic. It is a well known fact that for a substantial number of students the algebra presently taught in schools do not achieve its purpose, aiding students mathematical development and bring them the satisfaction of using a powerful tool for solving problems (Kieran, 2007; Loveless, 2008; MacGregor, 2004). These groups of students fail to use algebraic reasoning, for instance, to reformulate realistic problem tasks as equations or to apply appropriate technics to solve equations. Instead it seems some of them try to cope by learning procedures and disentangled rules to be remembered and applied in algebraic problem situations (MacGregor, 2004). Thus, algebra seems to be “a watershed in mathematics for many, both students and teachers” (Kilhamn & Röj-Lindberg, 2013, p. 306). My experience, as mathematics schoolteacher in compulsory school, is no exception. I have met a number of students striving with the school matter algebra – and felt the anguish of not being able to help. Therefore, every stone must be turned in the quest for more efficient approaches to algebra. Small as it may be, the aim of this thesis is to turn one of these stones.

For me personally algebra has been both applicable and helpful not least for my professional career, initially training to be a chemical engineer, later on becoming a mathematics teacher. However, some scholars (e.g., MacGregor, 2004; Noddings, 1994) claim that algebra, perceived the way I described above, as the manipulation of symbols, is of no consequence for the majority of people, especially with all the new technology making pen-and-paper methods redundant. If this is the case, we might not need to seek further to develop algebra teaching. On the other hand, there are a number of educational researchers (e.g., Kilpatrick, Swafford & Findell, 2001) asserting that algebra is very important for a growing number of people, not least *because* of the increased use of technology. Thus, it is no simple
matter knowing what to believe. One convincing, “hands-on”, argument to keep (re)searching, is the status of algebra as a "gatekeeper" in school mathematics. In many school systems algebra is considered to be of vital importance for student’s continuation to more advanced studies in mathematics, thereby affecting their chances for future education and employment (Cai & Knuth, 2011). Besides being regarded a screen instrument for higher education, algebra is credited a predictor for young students. Algebra learning is thus believed to be of great consequence for young students of elementary school (Loveless, 2008; Cai & Knuth, 2011; Watanabe, 2011). Besides reading, children’s mathematical achievement at the end of elementary school is considered to be a patent predictor of students’ final outcome (Kilpatrick et al., 2001). In order to meet and master the increasingly more complex mathematics of years to come, Blanton and Kaput (2011) in agreement with associate scholars (e.g., Schmittau, 2004; Carpenter et al., 2005), state that the mathematics of elementary school will have to focus more on the essential structures of the topic and provide children with a mathematical experience that goes beyond arithmetic and computational flow. To become mathematically prospering in later grades students need to have a profound understanding of the structural and general nature of mathematics (Blanton & Kaput, 2011). Viewing algebra from the perspective of the student, aiming for the more pleasant experiences it might bring, Schoenfeld (2008) states:

_The fundamental purpose of early algebra should be to provide students with a set of experiences that enable them to see mathematics—sometimes called the science of patterns—as something they can make sense of, and to provide them with the habits of mind that will support the use of the specific mathematical tools they will encounter when they study algebra. With the right kinds of experiences in early algebra, students will no longer find algebra to be a new and alien body of subject matter. Rather, they will find it to be the extension and codification of powerful modes of sense making that they have already encountered in their study of mathematics. (Schoenfeld, 2008, pp. 506-507)_

The combination of students’ experienced difficulties, often accompanied by their insufficient training and understanding in algebra, and the ever greater importance attributed to algebra, have directed the attention of policy makers and mathematics education researchers worldwide, toward algebra curriculum and algebra instruction (Cai & Knuth, 2011). This study aims at being part of this movement, contributing by increasing the knowledge about the algebraic contents of mathematics textbooks used in compulsory school.

Brizuela and Earnest (2008) pursue an interesting line of reasoning concerning mathematics and the conduct of language. Starting from the very beginning, and almost needless to say, the two scholars state that when children are introduced to spoken and written language, the introduction is spontaneous and happens in natural ways. When speaking to even the smallest child, grownups, most often, use conventional wording, vocabulary, and grammar. The same goes for written language. Children’s
books are, with very few exceptions, written in a conventional language using all the grammatical rules, sentence constructions and spelling regulations. These textbook texts do not try to mirror the children’s own language, but rather act the model from which the children can learn. Thus, one could say that children are introduced to all the complexity of language, both spoken and written, without mercy. Brizuela and Earnest (2008) claim that we would most certainly dismiss the idea of writing children’s books using invented spelling trying to reproduce baby talk. The researchers continue;

*However, when we move from written and oral language to the field of mathematics, these lessons learned are sometimes forgotten. We explicitly avoid complex mathematical terminology when speaking to students /.../ finding more simple, transitional language that we deem will be easier for them. /.../ we also avoid presenting students with notations that we deem would be too difficult for them to comprehend or adopt. (Brizuela & Earnest, 2008, p.274)*

In a survey on textbook research in mathematics education conducted by Fan, Zhu, and Miao (2013) the scholars acknowledge the important role of the mathematics textbook in teaching and learning. Mathematics, it seems, is a subject matter that, more than other school subjects, depend upon textbooks for teaching. The researchers claim that “the power of the textbook lies in their ability to serve as recourses which introduce readers to worlds which are not immediately obvious or cannot be experienced directly” (Fan et al., 2012, p. 635). As we shall see later, there is more than one reason for the emphasis on mathematics textbooks in school teaching. The conclusions drawn in the aforementioned survey are that textbook present mathematics contents, topics, and problem-solving inadequately. In addition, the studies reviewed, are said to reveal that there is a gap between textbooks and the intended curriculum (Fan et al., 2013).

1.2 Overarching purpose

The research results reflected upon in the previous section, in parallel with my own experience as a compulsory schoolteacher, have shown the significance of algebra in school education and therefore in educational research. Starting off with the assumption that algebra in fact is a difficult topic for many students and that algebra is a “gatekeeper” for higher mathematics education, there is, in spite of all previous research efforts, an urgent need to learn more about what might facilitate algebra development in students and how school instruction can be informed.

This thesis assumes that there is a beneficial “trajectory that one should take in instruction in order to ensure a meaningful learning of algebra” (Caspi & Sfard, 2012, p. 47, italics in original). This trajectory, which will be described in greater detail in the Theory chapter, is pictured as a stepwise progression involving consecutive developmental levels of increasing complexity. Furthermore, teachers’ instruction during lessons and the textbooks used in school are assumed to be means which may potentially support students’ algebraic development if these means are sequenced and/or organized in accordance
with the envisioned trajectory. However, the gap between mathematics textbooks and intended curricula reported on in the previous section, as well as the inadequate presentation of contents and topics, indicate that the mathematics textbooks do not constitute such a support. Thus, if the textbooks fail to support the beneficial trajectory more responsibility will be passed on to the teacher to fill in the gap.

A communicational theory introduced by Sfard (2001; 2007) is by some scholars (e.g., Newton, 2012; Caspi and Sfard, 2012) claimed to be an efficient means to study mathematics and/or mathematics learning. In this communicational theory mathematics is regarded as a discourse. From this follows that algebra also can be regarded as a discourse. This thesis takes on such a discursive perspective on algebra. Only a few research publications claiming to have a discursive perspective conceptualize mathematics itself as a discourse (Ryve, 2011). This thesis is one of them. Furthermore, the investigations performed in this study rest upon the assumption that both the texts of the mathematics textbooks and the talk of teachers’ instructions can be treated as discourses. Such an approach has been successful in others studies (Newton, 2012).

This study focuses on algebra, text in mathematics textbooks, and the talk of teachers’ lesson introductions. Algebra is, among various other things, about “unknowns”. These unknowns can appear, for instance in the form of symbols, or in the form of words uttered in talk or written in a text. This thesis omits the symbols and examines the words and phrases used to express these unknowns, both in the utterances of participating teachers and in the texts of mathematics textbooks. The overarching purpose of this thesis is to enhance the understanding of how classroom discourse supports the students’ learning of algebra.
2. BACKGROUND

This chapter starts with an account of a few definitions of algebra put forth within the educational research community. The traditions of algebra school instruction are described as well as two broad research orientations related to algebra: pre-algebra and early algebra. Subsequently, some empirical research results concerning algebra teaching and learning as well as research concerning the significance of the mathematics textbook for instruction are presented. Finally, the assumptions underlying this study are stated.

2.1 Defining algebra

Reviewing the field of educational research in search for an unequivocal definition of the notion of algebra is as vain as it is difficult. How scholars define algebra and what they deem to be a relevant focus vary considerably both within and across different research groups, not least due to cultural differences (Lins & Kaput, 2004; Kilpatrick et al., 2001). In addition, capturing all aspects of algebra is made even trickier because “what it is” is in constant alteration. Algebra develops both as a cultural artefact, incessantly including new symbols, and as an activity, effectuated by students engaging in the learning process. Therefore, there are and have been many different understandings of the characteristics of algebra, and different notions often pinpoint different aspects of the concept (Kaput, 2008; Kieran 2004a; Usiskin, 1999; Watanabe, 2011).

Thus, over the years, the nature of the subject matter algebra has been elaborated and somewhat reconsidered (Watanabe, 2011). Once being conceptualized almost exclusively as a topic focusing on manipulation of symbols and working out different kinds of equations and inequalities, the comprehension of algebra has expanded (Kaput, 2008; Mason, 2011). One past attempt to delineate algebra identified four conceptions of algebra; as generalized arithmetic, as a problem solving tool, as the study of relationships among quantities, and as the study of structures (Usiskin, 1999).

Yet another way of characterizing this topic is put forth by Kaput (2008) who register two distinguishing features of algebra and algebraic thinking. The first key aspect is creating generalisations and expressing these generalizations in conventional symbol systems of increasing formality. The second key aspect involves “reasoning with symbolic forms, including the syntactically guided manipulations of those symbolic forms” (Blanton & Kaput, 2011, p. 7).

The characteristics of algebra are somewhat differently put by Kieran (2004a) who fashion school algebra with the help of the activities that students typically engage in. The three categories offered for capturing the essence of algebra are the following; generational activities, transformational activities, and global meta-level activities. Generational activities entail forming algebraic expressions and equations, whereas transformational activities denote rule-based manipulation of these expressions. The global meta-level activities, for instance, problem solving, studying change, and analysing relationships, are of outmost importance for the meaning-building processes. For these activities algebra is
but one of many tools used. Nonetheless, Kieran (2004a) claims, that these meta-level activities are essential to all algebraic activities, because without them, all sense of purpose and meaning is lost.

Somewhat surprisingly, Kaput (2008) is the only one in these examples who explicitly mentions symbolism and the tampering with symbols in the context of defining algebra. Nevertheless, Kaput’s (2008) second key aspect and Kieran’s (2004a) third activity might be interpreted as referring to a somewhat analogous phenomena, conveying one’s thoughts and the mathematical activity as a whole. Such a meta-perspective associated with algebra is one of the themes of the present work. Perhaps the most striking commonality among the three scholars is the agreement that generalization of some sort is a distinguishing feature of algebra. The latter, that is, generalization – what it is and how it can be interpreted in the context of algebra - is a second theme of this thesis. The forthcoming text will elaborate on both of these themes and motivate why symbolism is moved to the rear.

2.2 Traditional school algebra

Mathematics instruction materials, for instance mathematics textbooks, tend to divide mathematical content into isolated topics (Carraher & Schliemann, 2007; Fan et al., 2013). Thus, when examining mathematics textbooks you might find that one chapter is labelled “Geometry”, another is called “Probability”, and yet another goes by the name of “Algebra”. Whether this classification is a result of historical teaching traditions or a reflection of the developers’ theory of how students learn mathematics is difficult to say. Nevertheless, research findings suggest that in mathematics, algebra is traditionally included and taught in a sequence of algebra-after-arithmetic (Kaput, 2008; Lins & Kaput, 2004; Cai & Knuth, 2011). Unquestionably, the story about algebra began with the science of number, that is, with arithmetic (Radford, 2001; Sfard & Linchevski, 1994; Sfard 1995). Historically, algebra grew out of the arithmetic field of knowledge and is therefore often perceived as a generalization of arithmetic or as arithmetic with letters, that is, as operations on generalized number symbolised by letters instead of operations on numbers (Carraher et al., 2006; Manson, 2011). This “natural” growth of algebra, as an offspring of arithmetic, might lead to the conclusion that in school instruction, algebraic knowledge “naturally” grows out of arithmetic knowledge. Sadly to say, this is not the case. On the contrary, both in-service teachers and prior research confirm that the introduction of algebra constitute an obstacle for large groups of students (Filloy & Rojano, 1989; Kieran et al., 1990; Loveless, 2008).

2.3 Pre-algebra

Attempting to address the issue of algebraic difficulty and find means to overcome the predicament many students face, most research findings reported on up to the early 1990s highlighted students’ difficulties and developmental constrains (Carraher & Schliemann, 2007; Kaput, 2008; Kilhamn & Röj-Lindberg, 2013). This orientation of interest, trying to extensively describe student errors and depict stages of cognitive development, thus focusing “sad stories”, is by some scholars called a pre-algebra approach (e.g., Herscovics & Kieran, 1980; Herscovics & Linchevski, 1994; Lins & Kaput,
From the pre-algebraic point of view, difficulties are attributed to the inherent differences between arithmetic and algebra (Carraher & Schliemann, 2007). Kilpatrick et al. (2001) state that “for most students, school algebra—with its symbolism, equation solving, and emphasis on relationships among quantities—seems in many ways to signal a break with number and arithmetic” (p. 255). Thus, algebra is by many regarded an essentially different and more advanced topic than arithmetic (Filloy & Rojano, 1989; Herscovics & Linchevski, 1994; Kilpatrick et al., 2001). Although adherents of this approach generally do not question the traditional curricula separation reported on above, the problems associated with the transition from arithmetic to algebra are by a growing number of scholars considered to be the likely result of this instructional sequencing rather than due to students’ deficient cognitive development (Carraher & Schliemann, 2007; Dougherty, 2008; Kieran, 2004b).

### 2.4 Early algebra

Research on algebra education reported on after the millennium shift has oriented itself somewhat differently than the pre-algebra approach and takes a more inclusive posture concerning arithmetic and algebra (Kaput, Carraher & Blanton, 2008). Adopting the Early Algebra perspective means zooming in on the kinship between arithmetic and algebra and acknowledge that “the boundary between arithmetic and algebra is not as distinct as often believed to be the case” (Cai & Knuth, 2011, p. ix). This latter-day rendering of algebra, by many denoted an Early Algebra approach, reflects the idea that “arithmetic has an inherently algebraic character” (Carraher et al., 2006, p. 89). From the Early Algebra perspective the problems that students experience with algebra are understood to primarily emerge from deficient introduction of arithmetic and elementary mathematics as a whole, not from algebra being inherently difficult in itself (Cai & Knuth, 2011; Carraher & Schliemann, 2007). Arithmetic is typically outlined as the science of number. Viewing arithmetic as a part of algebra is to say that arithmetic topics and examples can be seen as special instances of more general and abstract notions and ideas (Carraher & Schliemann, 2007). Therefore, in the implementation of curriculum, algebra is potentially accessible in parallel with arithmetic. Within the framework of Early Algebra, it is thought possible to develop algebraic understanding in the context of measurements and number work by deliberate generalization of the arithmetic concepts (Carraher & Schliemann, 2007). Russell, Schifter, and Bastable (2011) describe research-based activities that might be used in the classroom practice to highlight the connection between arithmetic and algebra. One of four key aspects these researchers recognize as important for illuminating the kinship between arithmetic and algebra is the behaviour of operations. Students can explicitly study operations when they are working with cubes. By joining cubes together students may illustrate addition. By subsequently change places of the sets of cubes it can be demonstrated that switching positions of addends does not change the sum.

Stating that arithmetic have an inherent algebraic character is not to say that all arithmetic procedures and concepts are plainly algebraic but that they are considered potentially so (Carraher & Schliemann, 2007). An Early Algebra approach suggests that student should be introduced to algebraic ways of thinking much earlier than commonly done today. Research findings presented by a number of schol-
ars (e.g., Brizuela & Schliemann, 2004; Carraher et al., 2006; Radford, 2012) support the idea that young students can make use of algebraic ideas often thought to lie beyond their reach due to the nature of mathematics and/or due to cognitive developmental constrains. The notion of quasi-variable, proposed by Fujii and Stephens (2001), might help to elucidate some ideas, underlying algebraic thinking. Quasi-variables, which are made up of known numbers, are commonly employed in Japanese elementary schools. These quasi-variables are used in numbers sentences to display that a mathematical relationship stays true even if the numbers that are being used are exchanged. Such a number sentence is the following, where 65 and 28 can be considered acting as quasi-variables:

\[ 65 - 28 + 28 = 65 \]

The early algebra start is, however, not to be understood as just presenting younger children with the same old school algebra in the same ways that grade 7 or grade 8 students meet algebra in most schools. Instead one has to expand the views of algebra and regard algebraic thinking as an integral part of all elementary mathematics, gradually developing into more formal algebraic conceptions and representations (Carraher et al., 2006; Kaput, 2008). The basic idea is to introduce the children to some of the important concepts of algebra, for instance variable as presented in the example above, early on so that they are acquainted with these concepts before they are introduced to formal ways of representing them (Lins & Kaput, 2004). Defining what Early Algebra is all about Schoenfeld (2008) state that “[t]he purpose of early algebra is to provide students with the kinds of sense-making experiences that will enable them [sic] to engage appropriately in algebraic thinking” (p. 482).

### 2.5 Teaching and learning algebra

“*The process of learning mathematics is largely a matter of engaging with ideas.* (Bastable & Schifter, 2008, p. 183)

An algebraified elementary mathematics is envisioned to “empower students, particularly by fostering a greater degree of generality in their thinking and an increased ability to communicate that generality” (Lins & Kaput, 2004, p. 58). Carraher, Schliemann, and Schwartz (2008) state that we are only beginning to understand the circumstances under which early algebra learning is promoted. Based on their own empirical research Bastable and Schifter (2008) assert that in classrooms of early algebra learning, mathematics “is conceived as much more than a sequence of facts and procedures to be memorized. Rather, mathematics is a realm of exploration, and doing mathematics is a social process: Children learn to actively and purposefully conjecture, revise ideas, offer proof, and argue mathematically” (p. 183). That is to say that the point of departure for instruction should be the mathematical ideas of the students and instruction should be organized to explicitly focus on these ideas (Bastable & Schifter, 2008).

There are many scholars claiming that children can develop a better understanding of mathematics if they are given the opportunity (e.g., Bastable & Schifter, 2008; Tierney & Monk; 2008; Carraher et al.
Sid 11

2006). For example, students’ awareness of numerical operations can be elevated by simply substituting small numbers for large in an operational sequence. The large numbers seem to help students focus on structure and reason about these structures, rather than just solve for an answer (Zazkis, 2001). Some scholars also assert that children are gifted with natural abilities for making and expressing generality. In the cultivation of algebraic reasoning, school education should therefore do its best to utilize these natural powers (Lins & Kaput, 2004; Manson, 2008). Teaching children about algebraic thinking in the context of numbers would be a first step of the process of “immersing them in the culture of algebra” (Lins & Kaput, 2004, p. 47, italics in original). As declared above this enculturation entail the idea that, rather than learning specified parts of algebra and the mastering of formal manipulation technics associated with algebra, it is essential that students first get acquainted with ways of expressing algebraic activity, and then become familiar with ways of reasoning from the different forms of expressions. For instance, studying the national curricula of Singapore (Cai, Ng, & Moyer, 2011), researchers found that in the earlier grades pictures and real life objects are used to model problem situations. From these examinations the scholars conclude that by using pictorial equation solving instead of symbolical representation to represent quantitative relations, students are provided an opportunity to make generalizations, learn to represent algebraic problems, and hence develop their algebraic ideas. In Figure 1, an example of such a pictorial solution to an assignment is given.

<table>
<thead>
<tr>
<th>Assignment:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha and Gamma shared € 500 between them.</td>
</tr>
<tr>
<td>Alpha received € 100 more than Gamma. How much money did Gamma receive?</td>
</tr>
<tr>
<td>2 units = € 500 – € 100 = € 400</td>
</tr>
<tr>
<td>1 unit = € 200</td>
</tr>
<tr>
<td>Gamma received € 200</td>
</tr>
</tbody>
</table>

![Figure 1. Pictorial equation solving (Cai et al., 2011, p. 33, adapted version)](image-url)
The teacher is very important when it comes to providing favourable conditions, that is, giving students the opportunity to learn. In Sweden, the teacher often initiates a mathematics lesson by presenting some content to be discussed and demonstrating the solution methods for the related tasks (Bergqvist et al., 2009). The teachers’ demonstrations of task solutions are often prepared in advance, thus containing tasks and contents chosen by the teacher (Bergqvist & Lithner, 2012). Furthermore, research conducted in Swedish schools indicate that instruction often focus on procedures rather than negotiating meaning and uncovering underlying mathematical ideas (Bergqvist et al., 2009; Löwing, 2004).

Additionally, referring to the research findings presented at the 12th ICMI study, Doerr (2004) argues for a strong relation between communication and understanding, and thus the importance of teachers explicitly engaging in reasoning in the mathematics classroom and using language to model mathematical meaning and thinking. For example, research studies concerning teachers’ conceptualization of algebra show that many of both pre-service and experienced teachers were unable to discuss the nature of algebra or articulate their own understandings of the subject matter (Johnson, 2001, referred to in Doerr, 2004). Many teachers also seem to hold equation solving to be the main issue of algebra teaching, thus emphasizing procedural skills. In addition, teachers at times tend to miss out on opportunities offered by arithmetic to generalize patterns since they interpret these assignments as problem solving tasks (Bishop & Stump, 2000, referred to in Doerr, 2004). In conclusion, in order to enhance students’ opportunities to learn and understand mathematics, teachers need to be aware of the importance and benefit of their own communicational skills. They must be offered opportunities to develop these skills, both during training and as teachers in practice.

2.6 Language and algebra learning

Numerous studies in mathematics education strongly reinforce the claim that language is very important for mathematics teaching and learning in the process of building understanding (Adler, 1999; Moschkovich, 2007; Radford, 2012; Wakefield, 2000). According to Drouhard & Teppo (2004), “teaching and learning of algebra can be enhanced by including an explicit focus on the language aspects of the subject” (p. 257). In the process of learning algebra early, natural language is seen as an important starting point. In early stages natural language should be used as a means to represent algebraic relations (Caharrer & Scliemann, 2007). Being able to use a known language makes it easier for young students to make sense of algebraic concepts and expressions as well as reasoning about them (Kaput, Carraher & Blanton, 2008; Lins & Kaput, 2004).

In her research review focusing language in mathematics teaching and learning, Schleppegrell (2010) discuss the difficulties teachers’ face trying to combine precision and clarity of what is being communicated with the need to aid initiation of students into mathematics discourse. Everyday language might be appropriate for communicating some mathematics concepts, but to enable students’ participation in more advanced mathematics contexts, language must be used with more precision and accuracy
(MacGregor, 2002). Halliday (1978), on the other hand, discussing how to introduce new concepts to children, claim that “the more informal talk goes on between the teacher and the learner around the concept, relating to it obliquely through all the modes of learning that are available in the context, the more help the learner is getting in mastering it” (p. 202). In conclusion, these findings send a double message to the teacher. To enable student learning, the teachers must try to combine the precision of the mathematical language with sense-making via informal every-day language.

Many studies have examined classroom discourse but the results vary. Examining the presence of six different knowledge structures (classification, description, principles, sequence, evaluation, and choice), Huang, Normandia, and Greer (2005) concluded that the talk of the teachers’ express a varied, high-level knowledge structure. Research conducted in Sweden claims contradictory that teachers use mathematics terminology ambiguously and that some teachers also fail to use language efficiently to describe and explain contents and/or procedure to students (Löwing, 2004). Imm and Stylianou (2012) identify and characterize three different types of mathematical classroom discourse: high, low, and hybrid. High discourse classrooms were characterized by dialogue and rich, inclusive, and purposeful mathematical conversations. This setting was most beneficial for participants’ construction of new ideas, as well as, for them to subsequently act on these ideas by challenging and validating them. Typically for a low-discourse classroom was the one-way “telling” communication primarily governed by the teacher. These classrooms were focusing on the teacher's mathematics and on mathematical procedures. The hybrid discourse consisted of a combination of conceptual and procedural understanding as well as emanating from both teacher’s and students’ mathematics. All these studies show that classroom discourse is a complex and important area of study.

2.7 The mathematics textbook

Beside the crucial role of the teacher and the organization of teaching, a basic assumption of this study is that mathematics textbooks play an important role in school education. Some research results, relevant for this assumption, are presented here.

It seems that teachers rely heavily on mathematical tasks drawn from written curricular materials for classroom instruction (Stein et al., 2007). Internationally, textbooks are the most frequent basis for mathematics instruction, used with 75% of the students in grade four, and with 77% of the grade eight students, on average (Mullins et al., 2012). The situation in Sweden is no different; the corresponding number for instruction drawn from curriculum materials is 89% in grade four and 97% in grade eight respectively (Mullis et al., 2012). There are also data contending that Swedish students devote more time than the international average working independently on tasks, while the teacher moves about in the classroom helping those who need assistance (Skolverket, 2008).

The central positioning of the textbook is an effect of many synergetic factors. In Sweden, one such factor is teachers’ inclination to perceive the textbook as a guarantee for meeting the knowledge requirements in the National curriculum documents (Englund, 1999). The textbook also provides a
structure for planning and presenting the subject area and helps organizing the students work during lessons (Bergqvist et al., 2009; Newton & Newton, 2007).

Thus, in mathematics classrooms of present day, students in many countries are primarily preoccupied working on tasks. Stein et al. (2007) claim that the tasks students work with determine how and what they learn about mathematics. A somewhat different way of putting it is to say, “students learn what they are given opportunities to learn” (Hiebert, 2003, s 10). However, research findings from the Swedish context suggest that in mathematics textbooks the focus on different competencies is unfairly distributed in favour of procedural fluency. Consequently, mathematics textbooks should be complemented with other materials or teaching technics, for instance, tuition by the teacher, to bring about activities focusing a broader spectrum of competencies (Bergqvist et al., 2009). Somewhat similar results were obtained in a study conducted in Australia. An analysis of grade 8 mathematics textbooks, revealed a predominance of calculation procedures with relatively few tasks and explanations to support conceptual understanding (Dole & Shield, 2008). A comparative study of mathematics textbooks in grade 7 in Japan and USA, showed that 81% of the Japanese textbooks were devoted to explaining solution procedure for worked examples, whereas the corresponding amount in the US textbooks were 36%. In addition, The US textbooks devoted 45% of their space to unsolved exercises and 19% to irrelevant illustrations. The corresponding amounts in the Japanese textbooks were 19% and 0% respectively (Mayer, Sims & Tajika, 1995).

In conclusion, the textbook is in many countries central in the mathematics classroom. Since it often focuses on procedures instead of conceptual understanding, it has to be combined by the teacher with other curriculum materials and activities to give the students the opportunity to develop a better understanding of mathematics.

To summarize the Background, there seem to be some agreement on how an early entrance on the road to algebra could be envisioned. Not surprisingly three features come to the front; the student, the teacher, and the classroom. Firstly, giving students the opportunity to scrutinize arithmetic and explicitly articulating the generalities of numbers, operations, and patterns, and progressively formalize these activities is important. Secondly, teachers need to be prepared and alert to identify and utilize generalizing opportunities as they arise in the everyday work. Finally, crucial for the enterprise of an early algebra approach is building a classroom culture that value these arithmetic scrutinizing activities and knows how to communicate (Bastable & Schifter, 2008; Blanton & Kaput, 2008; Lins & Kaput 2004). Schifter, Monk, Russell and Bastable (2008) call the challenges facing early algebra curriculum designers a “fortunate irony” since “by organizing lessons around ideas most salient to children, we create a classroom environment in which children are able to engage in activity that is much closer to the mathematician’s own practice-formulating, testing, and proving claims of generality” (p. 445).
3. OUTSET OF THE THESIS

*What we think algebra is has a huge bearing on how we approach it.*

*(Kaput, 2008, p. 8)*

It might be quite clear from what is written here and shown in other studies (e.g., Carraher et al., 2008; Kilhamn & Röj-Lindberg, 2013; Watanabe, 2011), that the demarcation lines between pre-algebra, early algebra, and algebra are not definite nor identical in different countries. As can be read from the quote above, what algebra is made out to be or how it is conceptualized is important for how it is understood, learned, and investigated. Thus, the framing of different algebraic research endeavours are bound to influence and shape both our conceptualization of algebraic thinking and how we envision the pathway of algebraic growth (Kieran, 2011). The approach regarding algebra taken in this thesis will be clearly stated in the chapter of Theory, but there are some basic premises that will be accounted for here.

In the present work, algebra is considered to reside within arithmetic and early mathematics, weaving all parts of the existing school curricula tightly together. Algebra is thus thought inherent in all kinds of word problems, in different topics, and also in various representational systems (Carraher et al., 2008). The representational systems acknowledged as essential for Early Algebra are arithmetical-algebraic notation, graphs, tables, and natural language (Carraher & Schliemann, 2007). It is the natural language representation system that is focused on in this thesis. The term “natural language” is here interpreted as the wording of the textbooks, that is, the text in itself, leaving out symbols, tables, graphs and so on. Natural language also include the teachers’ own wording when addressing the students during instruction.

In this thesis it is also assumed that arithmetic concepts entail a certain amount of generalities. These generalities are conceived a helpful way for closing in on algebra (Lins & Kaput, 2004). Arithmetic is, perhaps most commonly, delimited and understood as the science of numbers. In the present work, following Carraher and Schliemann (2007), arithmetic is reckoned to be the science of numbers, magnitudes, and quantities. The rationale for this broader definition lies in the argumentation concerning mathematics and science. Depending on ones attitude towards modelling, one might exclude magnitudes and quantities from the realm of mathematics and altogether attributed them to the field of science. However, since modelling *is* of great significance in elementary mathematics, it is justifiable to treat magnitudes and quantities as mathematical notions. Such arithmetic notions are for instance: $3; 169; 0.4; 5$ pieces; $-7^\circ; \frac{1}{2}$ litre. Trying to denominate these signs in a somewhat more general vocabulary would be to talk about numbers, digits, tenths, amount, degree, volume, and the likes. Also, in the early algebra curriculum, mathematics is not thought to be dominated by formal algebraic notation. On the contrary, such expressions are envisioned to be only gradually introduce.
4. THEORY

In this chapter, the discursive perspective on algebra is elaborated on. In the first part (section 4.1), the implications of this perspective for teaching and learning are described. In the second part (section 4.2), the theoretical means for closing in on the discursive constituents of interest for this thesis, which up-till now have been called unknowns, are accounted for. Finally, the specific purpose and the research questions are presented.

4.1 A commognitive theory

As previously stated, this thesis takes a discursive perspective on algebra. This perspective on algebra is embedded in a greater construct, a communicational theory mentioned earlier, or more accurately, a commognitive theory constructed and delineated by Sfard (2001, 2007, 2008). This whole section is a summary of central aspects of the commognitive theory, which act as a foundation for this present study. Unless otherwise mentioned, the writings in this section are based directly on Sfard (2001, 2007, 2008).

In her theory of commognition, Sfard takes a participationist view on learning. In the participationist view, development is conceived of as twofold; both as a modification of individual human behavior and as an alteration of patterned collective activities. These patterned collective activities are taken to be culturally and socially established specific forms of human activity governing how and what people are doing. At a given time, different sociocultural settings give rise to different human activity. Thus, human activity differs both between individuals and between collectives. In this perspective, the activity of the collective is considered to evolve prior to the activities of the individual. Human development is thus understood as a change in collective forms of human doing, not a change in single persons. In addition, the participationist’s position offers an explanation for the historical development and the alteration of human forms of life. This evolution is considered the outcome of two corresponding interrelated processes that complement each other; the process of individualization of the collective and the process of communalization of the individual. In the process of communalization, peoples personal forms of doing blend in with the collective forms of doing, thus subtly affecting these activities and, over time, inducing a continual change in these collective forms of activity. At the same time, the collective ways of doing are a role model for the individual’s activity. Individualization is the process of gradually taking over the collective forms of doing. This adoption of action, subsequently enables the individual to perform tasks such as cutting the dog’s fur, writing a speech or solving a mathematical problem, in a unique and personal fashion, entirely on her own. From one generation to the next, the activity range of the individual develops. The “mechanism” making this constant refinement, increasing sophistication and accumulation of complexity in actions possible, is the activity-mediating phenomenon known as discourse. This view has consequences for how one might look upon and understand the essential concerns of mathematical education such as thinking, learning, and of course mathematics itself. Before looking closer on these issues, the features of discourse will be elaborated on.
4.1.1 Distinguishing features of discourse

Discourse is here defined in accordance with Sfard (2008, p. 297), referring to a special type of communication that is “made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions”. The features making a certain discourse distinct are how *words* are used, what *visual mediators* are employed, which *narratives* are endorsed, and what *routines* can be identified. These features are all interrelated.

Word use is, amongst others, about learning new uses of former encountered word. For example the word “square”, which a person might have encountered in her daily life as a name for the market place, is used quite differently in school. When being initiated in the discourse of mathematics, small children are told that it is a name for a geometrical figure with four corners and four sides. Word use is also about learning to use completely new terms, for instance polynomial.

The visual mediators that participants of discourse use are means to coordinate their communication and identify the object of talk. Concrete objects, either actually seen or just imagined, often mediate colloquial discourses. Mathematical discourses, on the other hand, usually involve symbolic mediators, which are created specifically to make it possible to communicate in this particular form.

The statements the interlocutors of a specific discourse produce, spoken or written, to describe the objects, the relation between the objects or what activities are done with or by the objects are called narratives. These narratives can be taken to be true or false, that is, they are either endorsed or rejected. An example of an endorsed mathematical narrative is “multiplying two positive number yields a positive number”.

Finally, routines are the repetitive patterns that can be discerned in interlocutors’ actions. In mathematics, regularity of numerical calculations, for instance distributivity in multiplication, is one example of a routine. This kind of routine, making propositions about the objects of mathematics, is called object-level rules. Another kind of routine, which is much more implicit, is denominated meta-level rules and they are assertions about the discourse itself, not of its objects.

4.1.2 Thinking

In the above described theory of human development, *thinking*, although appearing to be an individual form of human doing, could only develop from patterned collective activity. That is to say, thinking is developed through a process of individualization of interpersonal communication. Thinking is thus an individualized form of communication, a communication with oneself. With this perspective, the often thought dichotomy between thinking and communicating, for instance disclosed in our talk about thoughts as being conveyed, is made redundant. To explicate this reconstructed cognitive conceptualization uniting thought and communication, Sfard combines the two words cognition and communication into a new concept denoted *commognition*. Commognition, is by Sfard defined as a
term encompassing both “thinking (individual cognition) and (interpersonal) communicating” (p. 296) As all other human doings, it has its roots in the historically and socioculturally developed practices of society. Therefore, commognition is bound to follow certain rules. In the same way as various games are played using different tools, regulated by different rules, so are there various types of commognition governed by distinct rules but also varying in topics, word use and means for communication. A community in which all actors know the rules and are able to play the game can be called a discourse. Differently put, discourse could be framed as commognition in which participating individuals master the rules of the communicational activity. Individuals lacking this ability will not be able to participate, and are thus excluded from the discourse. 

Being able to participate in different discourses is important for all human beings in order to be part of the commognitive community. The way an interlocutor implements the features of word use, mediators, narratives, and routines, in a given discourse will be ratified or rejected by other interlocutors participating in the communication. When trying to become a participant in a novel communicational activity, and develop a new discourse, language as well as context and social interactions will play a prominent role. An important incentive for learning, that is, changing one’s discursive ways, is a commognitive conflict.

4.1.3 Learning

Within the commognitive framework, learning is defined as; “the process of changing one’s discursive ways in a certain well-defined manner” (Sfard, 2001, p. 3, italics in original). There are two kinds of developmental changes; object-level and meta-level learning. Object-level learning is the result of a closer investigation of already known objects of talk. The new findings make it possible to express and endorse novel narratives about the known objects and construct new routines to suit the findings. Using the example with a square again, one might discover that, in addition to four corners and four sides, it also has four right angles. The learner is thereby introduced to two novel words connected to the known object, in addition to being made aware of some new actualities concerning it. This kind of learning brings about an expansion of the existing discourse, it is thereby an accumulative type of development.

Meta-level learning, on the other hand, entails a change of the rules of the game, the meta-rules. Encountering a new discourse, ruled by other terms than the rules the student has observed up until then, is a ground for a commognitive conflict. A commognitive conflict emerges when people participate in discourse differently, for instance use words differently. If they are to overcome this disparity, one or the other must change her ways of exerting discourse in order to make communication work. Communication is obstructed by the fact that interlocutors endorse contradicting narratives. The introduction of a new object of talk, in mathematics discourse, that is a new mathematical object, might induce a reevaluation of the prevailing mathematical discourse as a whole, not only of its objects. The reason for this is that the routines of the existing discourse are not applicable to the new discourse. For
example, when introducing negative numbers it can no longer be asserted that the order of magnitude between two numbers is preserved in the process of multiplication. In order to come to terms with the incommensurable discourses, one contradicting the other, the ‘old’ discourse must be subjected to the rules governing the new one, whereas the mathematical discourse itself expand and increase in complexity. Only an expert participant can initiate the necessary changes in meta-rules. In school, this expert is the teacher. Hence, the teacher has a very prominent role for the prospect of students individualizing discourse.

Children’s discourses differ from the typical school discourse in at least three ways; in its vocabulary, in the visual means which helps mediate communication, and in the meta-discursive rules which prescribe what proper, and inappropriate, discursive measures might be taken in a specific discourse. When teaching mathematics in school, the teacher’s task is to try to modify and extend the children’s existing discourses since the discourse to be learned must develop out of the discourses that the children already master. Entering school, the discourse that the children master is the discourse of their everyday life. Rather than building completely new discourses from scratch, the aim of learning mathematics is to transform these everyday spontaneously learned discourses in to a likeness of the mathematical discourses developed through history. Contrary to more conventional conceptions of the sequencing of learning, where students are thought to have a need for developing at least a partial understanding of a concept before they will be able to name or use it, the discursive approach envision it the other way around. That is to say that in the commognitive framework, learning is thought to start with the introduction of new names. Learning is then effectuated by the child taking advantage of learning techniques that have proven successful in the past, or differently put, using skills that worked when learning colloquial discourses. A new name, that is, a new signifier, can be inserted into known discursive templates of a colloquial discourse. These “try-out” insertions bring with them certain ontological messages making it possible to use the new signifier in suitable, or near suitable, discursive ways (Sfard, 2000). Sfard (2000) exemplifies the outset for such a try-out by using the invented signifier “krasnal”. If you have never encountered this word before but you hear it for the first time in the sentence The krasnal went to bed at ten, only one of the following sentences will make sense to you; Last night a krasnal watched TV or A krasnal was multiplied by four and then squared. In this way you will be able to start trying out what other sentences the word krasnal fits.

Mathematics learning is thus initiated by introducing new signifiers and using language on novel objects. It could be argued that it is really in the nature of things - in order to teach a new discourse, the teacher must, from the outset of the pursuit, talk about its objects. Thus, when the teacher introduces a new mathematical object, for instance “negative number”, it is likely that the students associate this object with the more well-known mathematical object “number” and initially make use of the discursive experience they have with this familiar object. That is to say that their previous encounter with the notion “number”, how it is talked about, what narratives and rules are applicable in communication about it, come in handy when getting acquainted with the new object “negative number”. Thanks to these former discursive experiences, and by using the new object in communication with others, trying
out what applications are appropriate and not, the learner gradually adjust her mathematical discourse to better fit the literal, historically developed mathematical discourse of school education.

When it comes to distinguishing between the different kinds of learning, the object-level and meta-level development mentioned earlier, the latter is harder to attain. Meta-level learning involves changes of the meta-rules of the discourse, and as such, this kind of development can only be instigated by the teacher. If anything, many of the meta-rules of mathematics can be regarded as conventions, rather than being inherently inevitable for mathematics. These conventions have their foundation in the historical evolvement of the notion in question and for newcomers to the discourse they are not obvious or easy to re-invent (Sfard, 2001). Drawing on the example of negative numbers again, an example of a meta-rule change will be given. Comparing the rule for multiplying two negative numbers yielding a positive number with the same procedure starting off with two positive numbers also ending up with a positive number, it is obvious that the rules for the operation on positive numbers (positive x positive -> positive), the rule endorsed up-till now, no longer holds. If it would hold, multiplying to negative numbers would yield a negative. This instance of a meta-rule change might not be self-evident for all students. As a consequence, students are often reluctant to accept such meta-rule alterations. Since these new rules are not governed by logical necessity they must be initiated by the teacher. Students who are willing to play the game and participate in effective communication will try to read the metadiscursive hints that the teacher gives, and imitate patterns in the discursive actions of the teacher and the other interlocutors. To argue for the necessity of this imitation, the following quote is offered; “if the children were not ready to follow the discursive lead of the grownups, they would never become able to communicate with other people” (Sfard, 2001, p. 13).

4.1.4 Mathematical objects

Every discourse is about “something”. Different discourses can be made distinct by the different objects they target. Such objects might either be imagined or actually seen, and they may either be concrete or discursive objects. Colloquial discourses, that is, discourses of everyday-life, are mostly mediated by images of concrete objects, which in turn might be imagined or visually observable. This goes for colloquial mathematical discourses as well. A school subject such as biology is about biological objects that, among other things, could be exemplified by plants. That is to say, it is about objects that are visually mediated by concrete material artifacts. As such, the discourse and its objects are separate entities, meaning that the objects of the communication exist independently of the discourse since they can be seen and touched. Mathematics, on the other hand, is a school subject about mathematical objects, for instance numbers. These objects differ considerably from the objects of biology. Take the number word ‘five’ for example. There is no single artifact to be pointed to. This signifier refer to an object which is a discursive construct, created in order to be able to talk about a certain amount of something, for instance pens, but it is also used in calculations without designating any real-life objects at all. Mathematical objects might be exemplified using visual means, but it could never be shown “in itself”. Or to put it differently, “mathematics begins where the tangible real-life objects end and
where reflection on our own discourse about these objects begins” (Sfard, 2008, p. 129, italics in original). Besides pointing to the elusive nature of mathematical objects, this statement highlights another discursive quality of mathematical discourse, namely that it, in itself, can be made the discursive object of mathematical discourse. Therefore, mathematical discourse can be envisioned as a structure of multiple levels, where each new layer might become the discursive object of yet another level. For instance the number words seven and two might on one elementary discursive level be talked about in the sentence “seven apples and two bananas”. In a higher discursive level the former might be stated as “the number of apples is five more than the number of bananas”, where the word “number” is a “metaname”, thus en elevated form, of seven and two.

A mathematical object is defined as a mathematical signifier together with its realization tree. In order to conceptualize what this definition really means, the terms signifier and realization tree must be scrutinized more closely. A signifier is a word, or a symbol, that function as a noun in communication between interlocutors. It is the means making it possible to talk about the mathematical object in question. Such a signifier is for instance the words two thirds. This signifier is interpreted, or more accurately, realized in the course of discursive activity (Sfard, 2012). Realizations are perceptually accessible, which genuine mathematical objects are not. Thus, a realization of the signifier two thirds could be the symbol \(\frac{2}{3}\), which would prove more useful in the context of calculation than the written or spoken words. The symbols \(\frac{2}{3}\), \(\frac{2}{3}\), \(\frac{2}{3}\), ... are all signifiers of the same mathematical object. This signifier of object could also be realized as a mark on a number line or as two of three equally sized parts of any physical object, for instance a cake or a drawn circle. However, the distinction between signifier and realization is relative. That is to say, two symbolic artifacts are often exchangeable in the roles as signifier and realization. Thus, the signifier \(\frac{2}{3}\) can be realized as \(\frac{2}{3}\) and vice versa. Thus, every realization can be viewed as a signifier, and this signifier has a realization tree of its own, and so forth. Consequently, by referring to a perceptually accessible object, the signifier is realized during the discourse activity either visually or vocally. Visual realizations can be verbal, iconic, concrete or gestural (Sfard, 2008). A realization tree is a “hierarchically organized set of all the realizations of the given signifier, together with the realizations of these realizations, as well as these latter realizations, and so forth” (Sfard, 2008, p. 301).

### 4.1.5 Algebraic objects

**Definition:** A signifier of an algebraic object is a noun or a noun phrase, serving as a noun, that signifies an unknown number, magnitude, or quantity.

An algebraic object is, in this thesis, perceived as a particular kind of mathematical object. That is, algebraic objects form a subsection within the group of mathematical objects. This study examines what signifiers are used to communicate about these algebraic objects. The definition of the signifiers of algebraic objects proposed in the present work has guided the analyses of the mathematics text-
books and of the teachers’ introduction talks. As will be described in the next section, algebra is understood to be a meta-discourse of arithmetic. Taking a meta-perspective on arithmetic means taking a meta-perspective on the science of numbers, magnitudes, and quantities (Carraher & Schliemann, 2007) as described in the outset of this thesis.

Founded in Sfard’s (2008) determination of signifiers as “words or symbols that function as nouns in utterances of discourse participants” (p.154) the aim of the definition is to pin down the words and phrases that can be interpreted as being “about” unknowns. In other words, this definition is designed to zoom in on the point in development when numbers are not only talked about in terms of seven, thirty-eight or \( \frac{2}{5} \) etc. but are talked about as for instance a number, the ratio or as the biggest area. The development of the algebraic discourse, delimited by the definition stated above, is assessed and analysed in comparison to the analytical framework, which will be described in the next passage.

### 4.2 Analytical framework

The analytical framework describes the theoretical basis for analyses in the present study. The framework is constructed within the boundaries of the commognitive theory. The corner stone for its construction can be found in a model issued by Sfard (1991; 1995) and Sfard and Linchevski (1994), aiming to conceptualize abstract mathematical thinking whilst linking individual mathematical growth to historical developments in mathematics. With a primary interest on the early stages of algebraic development, this model was further developed and operationalized by Caspi and Sfard (2012). In the present work, a section of this model is used for part of the analyses. In the following passage, both the model presented by Caspi & Sfard (2012) and the section used in the present study will be described.

Elementary school algebra, which is the mathematical contents focused in this study, is by Caspi and Sfard (2012) defined as a discourse, more specifically as a meta-discourse of arithmetic. Thus, algebra is in a sense viewed as in the Early algebra approach described in the Background, that is, as generalized arithmetic. Algebra is a discourse about a discourse, the arithmetic discourse. The word discourse is used in accordance with the commognitive theory reported on earlier in the text. Developmentally this meta-level evolvement entail changes of the rules of the game, that is, the meta-rules, making the discourse expand and gain in complexity (Sfard, 2012). The algebraic discourse starts with communication about processes, for instance an utterance like “add the numbers and find the answer”. This statement could, after some discursive development be phrased “what is the sum of the numbers”. These meta-level changes will be accounted for in greater detail in the following.

An endorsed narrative about arithmetic is that arithmetic is the science of numbers and calculations (Carraher & Schliemann, 2007). Therefore it could be claimed that arithmetic is about processes, actions, doing something with abstract entities such as numbers. Embracing the meta-discursive definition of algebra proposed above means turning to an algebraic stance regarding these activities, a stance where these actions, at a meta-level, are made into objects of discourse.
Trying to understand the change of discourse necessary to get from a posture of process to a posture of object, that is, a (first) meta-level, the sight is set on the model of abstract thought set-up by Sfard (1991; 1995) and Sfard and Linchevski (1994). In this model, the route of development and learning is delineated as a complicated process where two incompatible but complementary conceptions, an operational and a structural, of one and the same abstract notion interacts. An operational conception of a notion means perceiving it as a process, whereas a structural conception means viewing it as an object (Sfard, 1991). In this way, conveying 2 + 3 as a process would be viewing it as a calculation to be performed. Conveying the same expression as a signifier for the number five would equate viewing it as an object. This was a rather simple example but the same is true for an expression such like (5 + x). A person not initiated into the realm of unknowns, conveying this expression as a process, might find it impossible to proceed further, whereas a person being able to convey the same expression as an object, which in turn could be operated on, might have no problems at all (Sfard & Linchevski, 1994). “Seeing a mathematical entity as an object means being capable of referring to it as if it was a real thing – a static structure, existing somewhere in space and time. It also means being able to recognize the idea at ‘a glance’ and to manipulate it as a whole, without going in to details” (Sfard, 1991, p. 4).

The accomplishment of turning a process into an object, is a complicated endeavour entailing three stages; interiorization, condensation, and reification. In the first stage, the learner gets to know the mental operations of the processes by heart. Interiorization is achieved when a process constituting a number of operations, can be performed only in “the mind” without being actually executed. During the period of condensation, sets of operations constituting the process are “squeezed” into shorter units, thus making it possible to view the whole chains of operations as entireties. The first two stages are both progressive, quantitative changes whereas the final stage, the reification, is a qualitative alteration, a “quantum leap”, bringing with it the solidification of a process into a static structure, an object. This newly developed object can then be an input in yet another run-though of the three phases.

Consequently, mathematics, and thus algebra, might be conceptualized as a multi-layer structure where the same notions are conceived differently when viewed from less or more advanced mathematical positions, where operational conceptions precede structural conceptions (Sfard & Linchevski, 1994). A hierarchical structure of algebraic discourse arise due to mathematical discourse developing by annexing its own meta-discourse where each level but the initial is a discourse about the discourse comprised in the preceding level (Caspi & Sfard, 2012). Schematically, elementary algebra discourse development can be pictured as in Figure 2. Thus, the basic (meta-) level might for instance be the discursive level of number words like “seven” as described in the previously given example. A first discursive meta-level could be a about “number”.
Figure 2. Mathematics discourse development.

Evidently, the image in Figure 2 is not unique for algebra. All mathematics discourses are conceptualized as hierarchical structures. Developing algebraic discourse among students in school is envisioned as a route, at least in some extent, parallel to the historical evolvement of formalized algebra. That is to say, elementary algebraic discourse develops through deliberate regulation and objectification, and with the help of symbols. In order for students to become fully-fledged participants in algebraic discourse they must individualize the formalized algebraic discourse taught in school. As distinguishing as symbolization might be regarded for algebraic narratives, it is not a decisive means for expressing algebraic activity. It has been asserted that algebra goes back a long time. In the pre-symbolic algebra, algebraic statements were expressed verbally, that is, with words. Recognizing this fact it can be inferred that, if a rather complex algebraic statement expressed verbally can be transformed into a corresponding symbolic formula, then in turn, a fairly sophisticated symbolic expression could be phrased in words. As pointed out by Sfard and Linchevski (1994), “the history of algebra is not a history of symbols.” (p. 197)

One of many possible trajectories for algebraic discourse development is outlined in Figure 3. Here the spontaneous growth of children’s informal, everyday discourse constitutes the groundings for algebra instruction in school. Learning elementary algebra would equal elaborating this informal communicative strand in a formalizing process aiming to regulate and objectify the discourse alongside evolving a capacity for symbolism. Subsequently, the formalized version may go through additional advancement, still keeping in touch with the informal algebraic talk “that, in turn, may continue to develop in response to both the formal advancements and the learners’ needs and dispositions.” (Caspi & Sfard,
2012, p. 51). Such a trajectory would maintain continuity between the discourses already familiar to the student and discourses that the student still has to learn.

The hierarchical property of algebraic discourse development, as depicted by Caspi and Sfard (2012), is a five-level structure entailing both informal and formal strands. Grounded in previous empirical research and historical landmarks of algebraic evolution, the outline these scholars offer is claimed to reflect the possible growth of algebraic discourse, as well as indicating an eligible course of instruction for meaningful learning of algebra (Caspi & Sfard, 2012).

The model in its totality ranges from the very initial stages of spontaneous evolution of meta-arithmetic discourse, to the very advanced levels of modeling and exploration of functions. Of the five-level construct, the first three levels are labeled “constant value algebra”. At these levels, the signifiers of objects, whether expressed in words or symbols, are perceived to refer to specific numbers, either unknown or known, either sought or given. The two remaining levels have the common name “variable value algebra” since the discourses of both these levels deal with change and variation. At these levels, the objects of discourse are called variables and function. The area of interest for this thesis lies in the former partition, that is, the discourses dealing with constant value algebra. These three consecutive stages are said to depict initial algebraic growth. Since the textbook study aims to track down the first signs of a meta-arithmetic discourse, these first levels are the most pertinent. The time has now come to look a bit closer on the three initial stages of algebraic discourse development, the discourses of constant value algebra.

For the purpose of this thesis, a section of the major model of elementary algebra discourse development presented by Caspi and Sfard (2012) is rendered in Table 1. It is thus a simplified and reduced version of the original framework.
Table 1. Levels of elementary algebra discourse (Caspi & Sfard, 2012, pruned version)

<table>
<thead>
<tr>
<th>Level</th>
<th>Informal</th>
<th>Formal</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Signifier of object</td>
<td>Signifier of object</td>
</tr>
<tr>
<td>Level 3: Objectified</td>
<td>Complex noun clause used in an objectified way, as a description of the result of a specific calculation</td>
<td>Complex literal expression used as an object</td>
</tr>
<tr>
<td>Level 2: Granular</td>
<td>Complex noun clause used in the processual way, as a description of a specific calculation</td>
<td>Complex literal expression used as a prescription for a process</td>
</tr>
<tr>
<td>Level 1: Processual</td>
<td>Colloquial noun used as an unknown or a given number</td>
<td>Letter used as an unknown or a given</td>
</tr>
</tbody>
</table>

The educational materials, both the textbooks and the teachers’ instructional talks, are examined in comparison with the discursive progression outlined in the horizontal levels of the model in Table 1. The three stages constituting the section named “constant value algebra” are the levels of processual, granular and objectified discourse. Since the focus of this thesis is the written texts of the mathematical textbooks and the spoken words of the teachers, all symbolic representations, tables, graphs etc. are deliberately omitted. The emphasis of the forthcoming presentation is therefore solely on the informal stance of discourse.

According to the theory of algebraic development, at the first level, the processual stage, the focus is on numerical operations. Whether the calculations are presented in symbols or in words, the descriptions of the operations are listed in a linear order of execution. The equality symbol is often used as a “do something signal” as described by Kieran (1981). Informal phrases typically feature colloquial nouns used as unknown or given numbers. To exemplify an operational description on this level, mixing both informal and formal modes, the following is suggested; “add tree to n, divide by four, and subtract one”. The verbs used to express these processes make it an operational description.

At the granular level, the second stage of development, the emphasis is still on numerical operations, but the descriptions of these are no longer in the order of their execution. To effectuate this alteration the verbs of these descriptions are replaced with nouns. The verbs found on the processual level, for instance ‘add’, are replaced with nouns, in the example this would be ‘sum’. The resulting expressions, where these nouns in turn are operated on (by means of applying a verb) are called noun clauses. This replacement redirects the communicational focus to be about the result of an operation instead of being about the implementation of one or more processes. Noun clauses, where verbs have been replaced
can be regarded as “shortcuttings” of the chains of operations. These “shortcuttings”, which are called granules, can be understood as the results of assisting calculations. With their help the expression of an operational chain can be shortened and condensed. An informal instance of such a condensed sequence is the following “multiply the difference between n and two by five”. The verb “multiply” puts focus on a numerical operation, but the noun “difference” creates a shortcutting for the process of subtracting n and two.

The third, objectified level, represents a stage of being perfectly capable of participating in algebraic discourse. Informal complex noun clauses have the same discursive status as numbers and can act as a replacement of number, that is, as objects. These complex algebraic expressions can be part of any numerical operation, in the way numbers are, and they are also used to describe relations between objects. The following verbal description will serve as an example of the objectification on this level; “A product of a sum of two numbers and their difference is equal to the difference between the squares of these numbers” (Caspi & Sfard, 2012, p. 51). The example consists of one long noun phrase, placing it at the objectified level.

In the present work, the theoretical framework by Caspi & Sfard (2012) has served as a great inspiration in several ways. First, it inspired the work of crafting the definition of signifier of algebraic object used in the analyses. Second, a section of the framework was used in order to specify levels of metarithmic discourse. In addition, the overall structure of the pruned framework previously presented in Table 1, inspired the whole analytical process and made it possible to differentiate among the signifiers of algebraic objects. The time has now come to specify the purpose of this thesis and to specify the research questions that are guiding the two studies.
5. PURPOSE AND RESEARCH QUESTIONS

The overarching purpose of this thesis is to enhance the understanding of how classroom discourse supports the students’ learning of algebra. The issue of learning is examined through a focus on progression of algebraic discourse in mathematics textbooks. Furthermore, in order to study classroom discourse more broadly, the algebraic discourse of teachers is examined in relation to the algebraic discourse of textbooks.

Therefore, the specific research questions, designed to address the purpose of the study, are the following:

- How does the number of signifiers of algebraic objects differ between mathematics textbooks from different grade levels?
- How does the complexity of the wording of the signifiers of algebraic objects differ between mathematics textbooks from different grade levels?
- How does the complexity of the wording of the signifiers of algebraic objects of teachers’ instructional talks resemble or differ from the wording and progression of discourse in the mathematics textbook?
6. METHOD

To address the overarching purpose and answer the research questions, the contents of selected parts of the mathematics textbook for three different grade levels were examined. In addition, three teachers' lesson introduction talks were audio-recorded and transcribed. All texts were examined for signifiers of algebraic objects. The complexity of signifiers of algebraic objects were analysed according to the number of words constituting the signifier of algebraic object and whether the signifier of algebraic object was \textit{granule-like} or not.

6.1 Data collection

6.1.1 Selection of textbooks

The first two research questions concern mathematics textbooks from different grade levels and therefore textbooks for grade 2, 5, and 8 were examined. The selection of school years was made with the intention to examine books representing both a propagation in contents and level of knowledge. In grade 2 the students have learnt to read somewhat on their own, hence the textbooks of mathematics withhold some amount of text, not only pictures. In addition, since grade 2 students are just beginning to get acquainted with school mathematics, the very first embryo of meta-arithmetic (algebraic) structures is thought to occur in these books. If there is a successive increase in mathematics discourse complexity in the textbooks used in the nine year Swedish compulsory school, the textbook texts of grade 8 would be almost as complex as it gets. This was the motive for examining the mathematics textbooks of grade 8. Midway between grade 2 and grade 8 is grade 5, thus envisioned as a suitable halt along the assumed route of increasing discourse complexity.

One specific series of mathematics textbooks was chosen: Matte Direkt. This series has been the object of investigation in other studies and is a fairly common mathematics textbook in Swedish schools (Brändström, 2005). The particular textbooks chosen were Matte Direkt Safari 2A and 2B (Falck & Picetti, 2011a; 2011b) for grade 2, Matte Direkt Borgen 5A and 5B (Falck & Picetti, 2012a; 2012b) for grade 5 and, Matte Direkt 8 (Carlsson et al., 2012) for grade 8.

6.1.2 Selection of pages from the textbooks

A stratified random sample of a total of 24 pages from each grade level was selected. To be able to clarify the sampling procedure, the structure of the books is first described. All chapters start with a page describing the fundamental goals for the chapter in question. These goals are said to be formulated in accordance with the core content of the mathematics subject in the National Curriculum for the compulsory school in Sweden (Lgr11). Each chapter of the five textbooks is then divided in four distinct sections. The four sections of each chapter differ in their function in instruction and, to some extent, in
their level of difficulty. The first section of any chapter is a part comprising basic assignments intended to practise the depicted fundamental goals. Following this main partition of all chapters, there is a short test aiming to check what the student has learnt. This is thus the second section. If the test works out well, the student moves on to the third section, a section of more advanced assignments. Less prosperous students proceed to the fourth section containing additional tasks focusing on the fundamentals, in order to practise the basic contents some more. In all chapters there is also a fifth concluding section, which differs in contents in the three grades. The final section in all chapters in grade 2 deals with units of various sorts. In grade 5 the terminal part of each chapter consists of tasks that are meant to be challenging. In the end of each book, that is, in both 5A and 5B, there is a passage of tasks intended for repetition and rehearsal of the contents of all previous chapters. Also in the textbook of grade 8, each chapter have a finalizing part. The main contents of this section are assignments calling for an encompassing appliance of knowledge. There is also a minor partition of this section containing questions of multiple choice type and group discussion tasks. In addition, towards the end of the grade 8 textbook, there is a passage of challenging assignments.

To represent all different parts of the chapters, with their somewhat different levels of difficulty and various contents, pages were selected by stratified sampling. In particular, this choice was based on the overarching purpose of how classroom discourse supports students’ learning of algebra, focusing on progression of algebraic discourse in mathematics textbooks.

The two books of grade 2 (2A and 2B) were together regarded as one book. The same was done for the two books of grade 5 (5A and 5B). In grade 8 there was only one book. Thus, in the following there will only be three books mentioned, one for each grade. The total amount of pages in the textbook for grade 2 was 288 pages, for grade 5 there were 320 pages and for grade 8 a total of 304 pages.

Subgroups of pages within every chapter of each textbook were prepared. These groups were structured in accordance with the formerly presented sections. Thus, the pages of each chapter was separated into five subgroups, A-E. These groups are; A) pages with the common basics, B) test pages, C) pages with additional fundamentals, D) pages with more advanced mathematics, and finally, E) pages with diverse completing passages of each chapter. After calculating the number of pages in each group for all three textbooks, the percentage distribution among the five types of subgroups, in every textbook, could be calculated. This distribution was then used as the basis for sampling out pages in each textbook. For instance, in the textbook of grade 2, the distribution resulted in the following amount of pages for each group; A: 43% or 10 pages; B: 9% or 2 page; C: 15% or 4 pages; D: 17% or 4 pages, and finally E: 16% or 4 pages.

Since the analysis was conducted on running text, the sample only included pages that contained a minimum of two sentences of mathematical text. Pages that were excluded were, for example, first
pages describing the setup of a particular educational material, pages with tables of contents, and pages with answers to the tasks. The total amount of pages possible to analyse was 234 pages for grade 2; 274 for grade 5, and 209 pages for grade 8. Among these pages, 24 pages were randomly chosen for each textbook.

In order to make it possible to compare the number of signifiers of algebraic objects between textbooks, the number of signifiers found in the analysed pages were recalculated and extended to fit the total pages of each book. For instance in the grade 2 textbook the factor, F, used was calculated using the following formula:

\[ F = \left( \frac{s \times 234}{24} \right) \]

\( s \) being the number of signifiers found

234 being the number of possible pages to analyse

24 being the number of pages randomly chosen

6.1.3 Selection of teachers

The third research question concerns teachers' instructional talks and therefore three teachers in grade 8 were audio recorded. The reason for choosing grade 8 was an assumption that the teachers of grade 8 would engage in a more developed mathematical discourse when communicating with their students than the grade 2 or grade 5 teachers, thereby providing a better chance for the researcher to encounter signifiers of algebraic objects in their talk. Hence it was assumed that the teachers of grade 8 would engage in communication using a vocabulary containing “unknowns” and not only number words or specified quantities and magnitudes.

Four head-masters at four different schools in northern Sweden were contacted, who recommended teachers for participation. The recommended teachers were approached via an email explaining the study's aim, scope, and how it was to be conducted together with an invitation to the teacher to participate. Five teachers enrolled in the study but two later chose to leave, so the study includes three teachers at two different schools. The three enrolled teachers were visited at their respective school during mathematics lessons in which they had planned an introduction talk to the students. Two of the teachers were visited three times whereas one teacher was only visited one time, since no suitable time for a second visit could be found.

The teachers’ introduction talks were audio recorded in order to capture the exact, authentic wording of the teacher. Thanks to these recordings, the transcripts could be held to reflect the contents of the
talks as complete as it is possible, which in turn served as a good starting point for the analyses. During all the visits, the researcher stayed at the back of the classroom during the teachers’ introduction talks, not intervening or partaking in any way. The total amount of time recorded was 114 minutes.

6.2 Data analysis

In this thesis, the discursive construct of signifiers of algebraic objects, are examined. As described in the Theory chapter, the signifier of an algebraic object is defined as a noun, or a noun phrase, serving as a noun, that signifies an unknown number, magnitude or quantity. Only written text was analysed, which in the case of the teachers means the transcriptions of the lesson talks. Thus, all symbols, pictures, diagrams, graphs etc. were omitted.

The analysis of data essentially consists of two steps. In a first step of analyses, the texts of the mathematics textbooks and the transcribed teachers’ introduction talks were scrutinized in search for signifiers of algebraic objects. In a second step, the signifiers of algebraic objects found were counted and characterised according to complexity, in order to be able to answer the research questions, which address on the number of signifiers of algebraic objects and the complexity of these signifiers of algebraic objects.

6.2.1 The first step of the analysis

In the textbook study, the initial step was to copy all the selected pages from the three textbooks. The recordings of the teachers’ lesson introduction talks were transcribed before analysis.

Besides the first part, all parts of the procedure of analysis relate directly to parts of the definition of signifier of an algebraic object.

1. Mark all verbs.
2. Mark all nouns.
3. Exclude all nouns that do not relate to a number, magnitude or quantity.
4. Exclude all nouns that do not relate to an unknown.
5. For all remaining nouns, mark the longest noun phrase that includes the noun.
6. Exclude all noun phrases that do not relate to an unknown.

In order to utilise a reliable method of analysis, the signifiers of algebraic objects were singled out using the following procedure. First, all verbs were marked with a yellow text marker. The purpose of this action was to make it easier to find the nouns. All the encountered nouns were then colour coded using a green pen. In the subsequent stage, the marked nouns were rated a number, a magnitude, a
quantity, or none of the three types. For instance, nouns like chair or bike were rated as none of the three types, whereas a noun like ratio was rated a number, litre was rated a magnitude, and centimetre was rated a quantity. This ruling could only be done taking the context of the noun into account. For instance, the Swedish word “tal” can either denote a number or relate to a speech someone is about to make or has made. It is only in the former meaning the word can be taken to stand for a numerical unknown, that is, being interpreted as a signifier of an algebraic object. Nouns rated as none of the three types were excluded from further analyses. All the remaining nouns were then subjected to an expansion process, with the aim of finding the longest noun phrase in the text that included the given noun. This expansion process entailed a procedure in which the noun, if possible, were put into longer phrases, in a step-by-step fashion. By successively adding word after word from the text surrounding the noun, the phrase grew into longer and longer noun phrases. A noun phrase is a phrase that syntactically can function as a single noun. For example, the following noun phrase can be made up by gradually adding words to the noun “number”; a number which is smaller than one. Starting by adding “a” to the noun number result in the noun phrase “a number”, adding also “which is smaller” to the latter gives the noun phrase “a number which is smaller” and so on. When no further word could be added in order to create a longer noun phrase, the expansion stopped.

Next, the longest noun phrase produced from a noun was used in the further analyses. That is, the analyses focused on the most complex signifiers of algebraic objects, in order to be able to analyse the level of complexity, as described in the research questions. All noun phrases containing some kind of calculation symbol, for instance +, -, ÷, or ×, were excluded from the analyses, since the present study focuses on natural language. Number words were treated as nouns since they are a name when verbally expressed. The remaining noun phrases were then judged to be signifiers of algebraic objects or not according to the proposed definition of such a signifier, see also Table 2. Thus, in the process of finding longer noun phrases, sometimes the “unknown” noun came to be known. For instance, “five” (singled out in the step of finding nouns, but would be excluded in the step of dismissing all but unknowns) could be added to the noun “litre” (which is an unknown singled out in the step of finding nouns). The resulting noun phrase would be “five litres”, which is a known magnitude. Some nouns could not be built into a noun phrase involving more than the word itself. Those appointed to be signifiers of algebraic objects, both single nouns and noun phrases, were analysed further.

In the following, an example of the analytical procedure is given, the question in the end of the original task text is omitted.
Textbook text from grade 8 (my translation¹):

In a bacterial culture there are 10 000 bacteria. The number becomes double the size in one minute.

1. Initially the verbs were located and marked yellow:

In a bacterial culture there are 10 000 bacteria. The number becomes double the size in one minute.

2. All nouns were marked green:

In a bacterial culture there are 10 000 bacteria. The number becomes double the size in one minute.

3. All nouns not relating to numbers, magnitude or quantity were excluded by crossing them over:

In a bacterial culture there are 10 000 bacteria. The number becomes double the size in one minute.

4. All nouns not relating to an unknown were excluded by crossing them over:

In a bacterial culture there are 10 000 bacteria. The number becomes double the size in one minute.

5. For the remaining nouns, the longest noun phrase including the noun was marked by underlining:

In a bacterial culture there are 10 000 bacteria. The number becomes double the size in one minute.

6. All the noun phrases not relating to an unknown were excluded by crossing over them:

In a bacterial culture there are 10 000 bacteria. The number becomes double the size in one minute.

¹ I en bakteriedling finns 10 000 bakterier. Antalet blir dubbelt så stort på en minut.
In this example the remaining noun phrases relating to an unknown were; The number, and double the size.

6.2.2 The second step of the analysis

The signifiers of algebraic objects found in the first step were characterised according to the number of words constituting the signifier, henceforth called the signifier length.

Complexity of wording was measured regarding two different kinds of measures; firstly as signifier length, and secondly in terms of whether the signifier of algebraic object was granule-like or not.

Concerning signifier length, there are three measures; mean value of signifier length, and number of words, which is sub-divided in two measures. The latter category includes signifiers consisting of two words or more (signifier length \( \geq 2 \) words), and signifiers consisting of six words or more (signifier length \( \geq 6 \) words), respectively. The limitations of two and six words respectively, is set somewhat arbitrarily. The limitation of two words is set in order to differentiate between one word signifiers and all the rest. The limitation of six words is a consequence of six words being the median of the signifier lengths found, omitting one and two word signifiers. There is a measure for one word signifiers and for signifiers of two words or more, thus, six words or more is a mid-way measure among the words three words or longer.

Since the analyses that was performed in this study concern nouns and noun phrases only, and not whole sentences or texts excerpts, the framework proposed by Caspi and Sfard (2012) could only be used as an inspiration for the preparation of the analytical tool utilized in the second step. The analytical tool used is shown in Table 2.

Table 2 Analytical tool for signifier complexity

<table>
<thead>
<tr>
<th>Signifier of algebraic object</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Granule-like</td>
<td>Noun or noun phrase signifying an unknown, as in the result of an assisting calculation</td>
</tr>
<tr>
<td>Unknowns</td>
<td>Noun or noun phrase signifying an unknown number, magnitude or quantity</td>
</tr>
</tbody>
</table>
The signifiers of algebraic objects singled out in the first analytical step are all “unknowns”. The signifiers of algebraic objects of the first step were subjected to a second analytical step in order to deem if they also could be judged as granule-like signifiers of algebraic objects. Granule-like signifiers of algebraic object are signifiers that can be interpreted as the result of a calculation, for instance words like “sum”, “difference”, “product” etc.

In summary, the four types of complexity of signifiers of algebraic objects are: mean value of signifier length, signifier length of two words or more, signifier length of six words or more, and granule-like signifiers of algebraic objects.

6.3 Reliability

The concept of reliability is about the extent to which certain research findings can be replicated. In other words, it refers to the degree of consistency and stability in a research instrument across a range of settings, and if used by different researchers (Merriam, 2009). The more consistent and stable the research instrument is, and hence giving predictable and accurate results, the greater its reliability (Kumar, 2011; Wellington, 2000). This study is twofold, examining both textbook texts and teachers’ lesson introductions talks. The teachers’ introduction talks were audio recorded in order to capture the exact, authentic wording of the teacher. Thanks to these recordings the transcripts could be held to reflect the contents of the talks as complete as it is possible, which in turn served as a good starting point for the analyses. The reliability of both the textbook investigation and the teachers’ lesson talks examination is enhanced by letting one and the same researcher perform all analyses. The signifiers of algebraic objects were singled out using a written step-by-step procedure, which was designed in order to utilise a reliable method of analyses. Nevertheless, by letting one and the same person make all the analyses the inter-subjectivity might be affected. Trying to make up for this risk, the procedure for analysis has been thoroughly described.

6.4 Validity

The validity of the results cannot be guaranteed by following a prescribed procedure, no matter how detailed (Maxwell, 2005). Instead “validity refers to the degree to which a method, a test or a research tool actually measures what it is supposed to measure” (Wellington, 2000, p. 30). Accepting that reality is a human construct and that a researcher never can capture an objective “truth”, assessing validity becomes a question of deciding whether the data presented is credible or not (Merriam, 2009; Maxwell, 2005; Wellington, 2000). Thus, there is no universal means to ensure validity. The examined textbooks are public documents crafted to serve educational purposes, they are not produced for the sake of research. As such, they are pre-exiting and independent of the research endeavour, which in turn will grant them some advantages compared with other methods (Hatch, 2002; Merriam. 2009). They will not give way to the same limitations as for example interviews or observations whose out-
come might be affected by the moods of interviewees or the presence of the researcher (Merriam, 2009). Choosing one textbook series instead of another is, of course, a subjective act, which affect the outcome of the study. The rationale for choosing the series of Matte Direkt was that it had been previously investigated and it is rather common in Swedish schools. Thus, the book series seems a representative means for Swedish school instruction. The stratified sample procedure was a measure taken to pursue the ambition of keeping close to the purpose and research questions of this thesis. The definition, and step-by-step analytical procedure singling out signifiers of algebraic objects, as well as the analytical tool for differentiating discursive levels, were all inspired by the theoretical framework proposed by Caspi and Sfard (2012). Since there was signifiers found, which in turn could be differentiated to some degree, results talk in favour for the validity of this study.

The participating teachers may have been affected by my presences in the classroom. Having presented the general research set-up of my study, the teachers willingly participated. If anything, my presence may have affected the instruction talks positively, meaning that the teacher might have paid more attention to the lesson talk, granting it greater importance. Such an increased interest would not affect the outcome of this part of the study negative, on the contrary.
7. RESULTS

In this chapter the results from the analysis are presented. The results are arranged in conformity with the three research questions raised in connection to the description of the purpose of the thesis. First, an account is given of the number of signifiers of algebraic objects in the three different mathematics textbooks for grades 2, 5 and 8 together with a description of their complexity. Thereafter, the signifiers of algebraic objects of the teachers’ lesson introduction talks in grade 8 are presented and compared with the results obtained in the three textbook studies.

7.1 How does the number of signifiers of algebraic objects differ between mathematics textbooks from different grade levels?

To start with, in the texts analysed there are signifiers of algebraic objects. In Table 3, the number of signifiers of algebraic objects in the textbooks of the different grades is shown. Out of the twenty-four pages chosen in each of the three textbooks, all pages in both grade 8 and grade 5 contained at least one signifier of an algebraic object. In grade 2, however, seven of the chosen pages carried no signifier of algebraic object at all.

Table 3. Number of signifiers of algebraic objects in mathematics textbooks for different grades

<table>
<thead>
<tr>
<th></th>
<th>Number of signifiers of algebraic objects in analysed pages</th>
<th>Number of signifiers of algebraic objects per textbook</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>One word signifier</td>
<td>Signifier ≥ two words</td>
</tr>
<tr>
<td>Grade 2</td>
<td>4</td>
<td>11</td>
</tr>
<tr>
<td>Grade 5</td>
<td>64</td>
<td>78</td>
</tr>
<tr>
<td>Grade 8</td>
<td>71</td>
<td>118</td>
</tr>
</tbody>
</table>

* 24 pages per textbook were analysed

** The total amount of pages possible to analyse was 234 pages for grade 2; 274 for grade 5, and 209 pages for grade 8.
As can be seen in Table 3 and Figure 4 there is a clear increase of signifiers of algebraic objects between grade 2 and 8. The greatest increase occurs between grade 2 and 5. The total number of signifiers of algebraic objects recognized in grade 2 constitutes approximately 10% of the total number in grade 5. The subsequent increase from grade 5 to grade 8 is very modest, almost negligible.

![Figure 4. Number of signifiers of algebraic objects in mathematics textbooks for different grades.](image)

To summarize, there were signifiers of algebraic objects in all three textbooks. The number of signifiers of algebraic objects increased between grade 2 and grade 8, with the greatest increase appearing between grade 2 and grade 5. The total number of signifiers of algebraic objects in grade 2 was approximately 1/10 of the number in grade 5. Between grade 5 and grade 8 the increase of signifiers of algebraic objects was modest.

7.2 How does the complexity of the wording of the signifiers of algebraic objects differ between mathematics textbooks from different grade levels?

7.2.1 Length of signifiers of algebraic objects

Exploring complexity in terms of signifier length, Table 4 shows all the categories found. None of the signifiers of algebraic objects encountered in the mathematics textbook of grade 2 was longer than five
words. Both of these five-word signifiers of algebraic objects were the phrase “the number ahead and the number after” (5 words in Swedish²).

Table 4. Number of signifiers of algebraic objects of different signifier length, and mean value of signifier length

<table>
<thead>
<tr>
<th>Number of words per signifier</th>
<th>Number of signifiers of algebraic objects of a specific length</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>One</td>
</tr>
<tr>
<td>Grade 2</td>
<td>4</td>
</tr>
<tr>
<td>Grade 5</td>
<td>64</td>
</tr>
<tr>
<td>Grade 8</td>
<td>71</td>
</tr>
<tr>
<td>Teachers</td>
<td>23</td>
</tr>
</tbody>
</table>

Furthermore, Table 4 presents the mean value of signifier length. This complexity measure shows that the average length of the signifiers of algebraic objects found in different textbooks increase steadily between grade 2 and 8.

Figure 5 shows the percentage distribution of signifiers of algebraic objects of different signifier length. In grade 2 and grade 5, signifier length of one and three words are the most common. In grade 5, one word signifiers represent almost half of all signifiers of algebraic objects found. In grade 8, approximately 60 % of all signifiers of algebraic objects found consist of one or two words. Furthermore, the graphs show that a signifier length between four and seven words are just about as common in grade 5

² In Swedish, “talet före och talet efter”
as in grade 8. However, a signifier length of eight words is more common in grade 8 than in grade 5. An example of such an eight word signifier is, “the area of a circle with the diameter 12 cm”

Figure 5. Number of signifiers of algebraic objects with a specific length, percentage of total.

In Table 5, the distribution between one word signifiers of algebraic objects and signifiers of algebraic objects containing two words or more is given. Comparing signifier complexity in terms of signifier length equal to or exceeding two words, the textbook of grade 2 has the highest amount of all three grades. In the grade 5 textbook, the distribution between one word signifiers and signifiers of algebraic objects of a signifier length equal to or exceeding two words is just about even, whereas in the grade 8 textbook the amount of signifiers of algebraic objects of a signifier length equal to or exceeding two words, exceed the amount of one word signifiers.

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3 In Swedish, arean av en cirkel med diametern 12 cm (12 is treated as one word)
Table 5. Complexity, signifier length ≥ two words

<table>
<thead>
<tr>
<th>Type of signifiers</th>
<th>Amount of total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>One word signifie</td>
</tr>
<tr>
<td>Grade 2</td>
<td>27 %</td>
</tr>
<tr>
<td>Grade 5</td>
<td>45 %</td>
</tr>
<tr>
<td>Grade 8</td>
<td>38 %</td>
</tr>
<tr>
<td>Teacher</td>
<td>25 %</td>
</tr>
</tbody>
</table>

Table 6 shows the signifier complexity measured as signifier length equal to or exceeding six words as well as complexity in terms of granule-like signifiers of algebraic objects. In the grade 2 textbook there were no signifiers of algebraic objects of a signifier length equalling or exceeding six words. Also in the grade 5 and grade 8 textbooks the majority of signifiers of algebraic objects are of a signifier length shorter than six words. The increase in the amount of signifiers of algebraic objects of a signifier length equal to or exceeding six words is 11 percentage points between grade 2 and grade 5, and 6 percentage points between grade 5 and grade 8.

Table 6. Complexity, signifier length ≥ six words and granule-like

<table>
<thead>
<tr>
<th>Signifier length &lt; six words</th>
<th>Signifier length ≥ six words</th>
<th>Granule-like</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of signifiers’</td>
<td>Amount of total</td>
<td>Number of signifiers’</td>
</tr>
<tr>
<td>Grade 2</td>
<td>15</td>
<td>100 %</td>
</tr>
<tr>
<td>Grade 5</td>
<td>127</td>
<td>89 %</td>
</tr>
<tr>
<td>Grade 8</td>
<td>157</td>
<td>83 %</td>
</tr>
<tr>
<td>Teachers</td>
<td>84</td>
<td>92 %</td>
</tr>
</tbody>
</table>

** 24 pages per textbook were analysed

**per textbook, The total amount of pages possible to analyse was 234 pages for grade 2; 274 for grade 5, and 209 pages for grade 8

***In seven lessons
Summing up this section, the average length of signifiers of algebraic objects in mathematics textbooks increases between grade 2 and grade 8. The amount of signifiers of algebraic objects of a signifier length equal to or exceeding two words decreased between grade 2 and 5, but increased again in the grade 8 textbook. Consequently, the amount of one word signifiers of algebraic objects increased between grade 2 and 5, to decrease in grade 8. The signifier length in the grade 2 textbook did not qualify as complex, in terms of signifiers equal to or exceeding six words, meaning there were no such signifiers of algebraic objects. Between the grade 5 and grade 8 textbooks, the amount of signifiers of algebraic objects of a length equal to or exceeding six words, increased from 11 % to 17 %. The amount of signifiers of algebraic objects of a signifier length of four to seven words is about the same in grade 5 and 8. In grade 8 there are also a number of signifiers of algebraic objects of a greater length than seven words.

7.2.2 Tendencies toward meta-level discourse development

In Table 5, the number of signifiers of algebraic objects recognized as granule-like are given. The grade 2 mathematics textbooks contained no granule-like signifiers of algebraic objects.

When comparing the pages analysed in the mathematics textbooks of grade 5 and 8, respectively, the number of granule-like signifiers of algebraic objects in grade 8 was just about twice the corresponding number in grade 5. However, comparing the amount of granule-like signifiers of algebraic objects, Table 5 show that the difference between grade 5 and 8 is 2 percentage points.

The granule-like signifiers of algebraic objects in grade 5 were all about mean value, which was a signifier taken to be an expression for the result of an assisting calculation. In the mathematics textbook of grade 8 all but one of the granule-like signifiers of algebraic objects were made up by the nouns “sum”, “quotient” and “product”. The remaining signifier was about mean value.

In summary, there were no granule-like signifiers of algebraic objects in the grade 2 mathematics textbook. In grade 5 the amount of granule-like signifiers of algebraic objects was 4 %. In grade the grade 8 the amount of granule-like signifiers of algebraic objects had risen and made up 6 % of the total number of signifiers of algebraic objects.

7.3 How does the complexity of the wording of the signifiers of algebraic objects of teachers’ instructional talks resemble or from the wording and progression of discourse in the mathematics textbook?

In Table 4 there is a summary of the number of signifiers of algebraic objects of both the mathematics textbooks and the teachers’ introduction talks. In the 114 minutes of recorded teachers’ introduction
talks a total of 91 signifiers of algebraic objects where recognized. The mean value of signifier length is also given in Table 4. The average length of teachers’ signifiers of object, 2.8 words per signifier, were shorter than the corresponding measure in any of the mathematics textbooks examined.

Signifier complexity measured as signifier length equal to or exceeding two words is shown in Table 5. From the numbers presented it can be inferred that the teachers’ talks contained just about the same amount of signifiers of a signifier length equaling or exceeding two words as the grade 2 mathematics textbook, approximately three-quarters of the total.

Table 6 shows that the amount of teacher talks that contained signifiers consisting of six words or more was 8 %, an amount lower than both the amounts in the grade 5 and the grade 8 mathematics textbooks.

The amount of granule-like signifiers of algebraic objects in the talks of the teachers is also shown in Table 6. Two granule-like signifiers of algebraic objects were recognized in the teachers’ introduction talks, which make up 2 % of the total amount of signifiers found in those talks. Comparing the amount of granule-like signifiers of algebraic objects of teachers’ talks with the corresponding amounts found in the mathematics textbooks, Table 6 show that, since there are granule-like signifiers of algebraic objects in the teachers’ wordings, the amount of granule-like signifiers of algebraic objects is greater than in grade 2, but smaller than in grade 5.

The two granule-like signifiers of algebraic objects that were recognized during the seven lessons of introduction talks were both constituted of a one word signifier of algebraic objects, ratio, and were uttered in one and the same introduction talk.

Summing up, the mean value of signifier length in the teachers’ introduction talks is the lowest of the examined discourses. The amount of one word signifiers of algebraic objects and signifiers of algebraic objects of a signifier length equal to or exceeding two words in teachers’ talks are approximately the same as the corresponding amount in the grade 2 mathematics textbook. Comparing signifier complexity measured as signifier length equal to or exceeding six words, the amount of signifiers of a signifier length equal to or exceeding six words in the teachers’ introduction talks falls in-between the corresponding measures of the grade 2 and the grade 5 mathematics textbooks. There are some granule-like signifiers of algebraic objects in the teachers’ lesson talk. Compared with the mathematics three mathematics textbook the number is greater than in grade 2, but it is smaller than in grade 5.
8. CONCLUSIONS AND DISCUSSION

The overarching purpose of this thesis has been to enhance the understanding of how classroom discourse supports the students’ learning of algebra. The issue of learning was examined by focusing on the progression of algebraic discourse in mathematics textbooks, and examining the algebraic discourse of teachers in relation to the algebraic discourse of mathematics textbooks. In the first part of this chapter, the research questions of the study are answered. In the second part, the results concerning the two discursive manifestations, mathematics textbooks and teachers lesson talks, are discussed and some proposals for future studies are advanced.

8.1 Answers to the research questions

8.1.1 How does the number of signifiers of algebraic objects differ between mathematics textbooks from different grade levels?

First of all, there were signifiers of algebraic objects in all three mathematics textbooks. The number of signifiers of algebraic objects increased between grade 2 and grade 8, with the greatest increase appearing between grade 2 and grade 5. The number of signifiers of algebraic objects raised from 146 to 1621 per mathematics textbook between grade 2 and grade 5. The total number of signifiers of algebraic objects in grade 2 was thus approximately 1/10 of the number in grade 5. Between grade 5 and grade 8 the increase of signifiers of algebraic objects was modest. The number of signifiers of algebraic objects raised from 1621 to 1646 per mathematics textbook between grade 5 and 8.

8.1.2 How does the complexity of the wording of the signifiers of algebraic objects differ between mathematics textbooks from different grade levels?

The four types of complexity of signifiers of algebraic objects measured were: mean value of signifier length, signifier length of two words or more, signifier length of six word or more, and granule-like signifiers of algebraic objects. The average length of signifiers of algebraic objects in mathematics textbooks increased between grade 2 and grade 8. The amount of signifiers of algebraic objects of a signifier length equal to or exceeding two words decreased between grade 2 and grade 5, to increase again in the grade 8 mathematics textbook. The length in the grade 2 mathematics textbook did not qualify as complex, in terms of signifiers of algebraic objects being equal to or exceeding six words, meaning there were no such signifiers of algebraic objects. Between the grade 5 and grade 8 mathematics textbooks the amount of signifiers of algebraic objects of a length equal to or exceeding six word increased from 11 % to 17 %. Furthermore, there were no granule-like signifiers of algebraic objects in the grade 2 mathematics textbook. In grade 5 the amount of granule-like signifiers of algebraic objects was 4 %. In grade 8, the amount of granule-like signifiers of algebraic objects had risen and made up 6 % of the total number of signifiers of algebraic objects.
8.1.3 How does the complexity of the wording of the signifiers of algebraic objects of teachers’ instructional talks resemble or differ from the wording and progression of discourse in the mathematics textbook?

The mean value of signifier length in the teachers’ introduction talks is the lowest of the examined discourses, 2.8 words per signifier. The signifiers of algebraic objects of a signifier length equal to or exceeding two words in teachers’ talks are just about the same as the corresponding amount in the grade 2 mathematics textbook, approximately $\frac{3}{4}$ of the total amount of signifiers of algebraic objects. Comparing signifier of algebraic object complexity measured as signifier length equal to or exceeding six words, the amount of signifiers of algebraic objects with a signifier length equal to or exceeding six words in the teachers’ introduction talks falls in-between the corresponding measures of the grade 2 and the grade 5 mathematics textbooks, thus being 8% of the total amount of signifiers of algebraic objects found.

8.2 Discussion and future research

In the following, the implications of the answers to the three research questions will be discussed in some depth. First, the results concerning teacher discourse are reflected on. Second, results related to the discourse of the mathematics textbooks are discussed.

8.2.1 Teacher discourse

I choose to start off the discussion with the last question, beginning with the very first analytical step leading to a conclusion, the transcription phase. Transcribing teachers’ lesson talk is no simple task since these talks are not a steady flow of words in the same way as a written text. Due to students interrupting the teacher, and, I think as a part of the pedagogical art, teachers often repeat parts of what they just said, making these talks rather fragmentary and somewhat difficult to transfer into written text, and even harder to analyse. Bearing this in mind when interpreting the results, and what one can conclude from them, there might be a number of reasons for the comparatively low degree of complexity in teachers’ talk.

The degree of complexity might be a deliberate choice by the teacher. Keeping the natural language as close as possible to the everyday ways of communicating, ways that students are familiar with, is reported beneficial for representing algebraic relations (Carraher & Schliemann, 2007) and must be the point of departure for teaching in school (Sfard, 2001; 2008). With the support of colloquial discourses, the teacher can help students try out, and subsequently learn, what a new signifier is all about, using known templates (Sfard, 2001; 2005). Like Halliday (1978), teachers might argue that informal language is the means with which concepts are built and understood. However, sticking to this mode of communication is counterproductive in the long run (MacGregor, 2002).
In order to prepare students for participation in more advanced mathematics discourses, teachers need to use a more advanced mathematics vocabulary, even though this may constitute a dilemma for the teacher striving to “ease” the way and assist all students (Brizuela & Earnest, 2008; MacGregor, 2002; Schleppegrell, 2010). Avoiding being explicit concerning new signifiers may result in ambiguity in word use and in the long run cause misunderstanding (Löwing, 2004; Sfard, 2001). It seems that students find mathematics vocabulary difficult (Schaftel, Belton-Kocher, Glasnapp & Poggio, 2006), the more reason to explicitly teach students mathematics vocabulary.

According to the commognitive theory, the teacher must be the one initiating the discursive change by introducing new signifiers (Sfard, 2001, 2008). Children do learn “the hard way”, that is to say, they learn even though, or perhaps thanks to, being introduced to talk and writings not subjected to any grammatical, or other, difficulty reducing measures (Brizuela & Earnest, 2008). If the more advanced meta-level signifiers of algebraic objects, in this thesis called granule-like signifiers of algebraic objects, were an explicit focus of discursive activity it can be argued that they would be a frequent ingredient in classroom communication, at least matching the amount in the mathematics textbook of grade 8. It seems they are not. What words are used instead and what are the rationales for doing so? This, I think, would be an interesting research question to answer in a future study.

Research findings point to the great significance and benefits of articulating one’s own understanding using mathematics vocabulary in order to uncover the underlying mathematical ideas and model mathematical meaning and thinking (e.g., Löwing, 2004; Silver & Smith, 1996; Zack & Graves, 2001). The mean value of the signifiers’ length in teachers’ talks being the shortest compared with the mathematics textbooks, in addition to a comparatively low amount of granule-like signifiers of algebraic objects, indicate that the contents of classroom discourse does not reflect the contents of the grade 8 mathematics textbook, in terms of signifiers of algebraic objects being present. Thus, it could be inferred that the potential of the mathematics textbook is not backed up in instruction. And more to the point, if these signifiers of the grade 8 mathematics textbook, with a comparatively greater difficulty are not made a specific focus of classroom communication, what are the consequences? Will this result in students ending up having difficulty doing algebra? Examining how the implicit mathematical and algebraic contents and structures are illuminated in the classroom discourse of different grade levels is of outmost importance in order to answer these questions and thus a mission for future research.

8.2.2 Mathematics textbooks

Discussing the findings from the mathematics textbooks in some greater detail I think it is convenient to start with the grade 2 mathematics textbook. The total amount of signifiers found in the grade 2 mathematics textbook was low compared with the two other books. Adding the fact that out of the 24 analysed pages, seven had to be dismissed because they contained no signifiers of algebraic objects at all, indicate that the signifiers of algebraic objects are scattered and not to be found on every page in
the mathematics textbook. Even before the stratified sampling procedure some pages were dismissed since they had no words at all, just pictures and/or numbers in tasks. No such dismissals had to be made in the grade 5 or grade 8 mathematics textbooks. These findings are interpreted as evidence that the written dimension of discourse lingers in a mathematics of known numbers. The few signifiers of algebraic objects found are indeed a sign of a first embryo of an algebraic or meta-arithmetic discourse.

The great increase of signifiers of algebraic objects between grade 2 and 5 indicates that between these grade levels, a great number of new unknowns are presented to the students. Research conducted in Sweden (2004), indicates that in grade 5, students lose the incentive to learn mathematics. Why is that so? Is it in fact a result of students starting to lose the grip of what mathematics is all about? If the meaning and understanding of these new signifiers of algebraic objects are not made an explicit focus of classroom discourse and students are left too much on their own, working in the mathematics textbook, how are they supposed to cope? If the mathematics textbook texts are not designed to help students build the appropriate understanding, due to inadequate presentation of the mathematics content (Fan et al., 2013) the teacher is left with this assignment. The focus is thus on the teacher once more. To facilitate students’ algebraic learning in the light of an Early Algebra approach, making these inherent algebraic structures an explicit focus of classroom discourse is of great importance. In a commognitive perspective the teacher need to be the one initiating a discursive change, that is, learning. Changing students’ discursive ways means altering and/or extending the rules, narratives and word use of the existing discourse. This is accomplished by the teacher, starting to talk about the new signifier, the unknown, telling about it in a communicational act where students discursive ways develop through a process of “individualizing the collective”. At the same time, the collective, in this case the classroom discourses, goes through the process of “communalization of the individual”. In short, there is a mutual exchange. What discursive activities are involved in this mutual exchange? How are the new signifiers of algebraic objects talked about by teachers and by students? How does the changes of rules and narrative go about? Differently put, how do teachers and students talk and reason about these new unknowns or the inherent algebraic structures in mathematics classrooms? How do they build meaning and understanding? How are new unknowns introduced by the teacher? What aspects are illuminated and what aspects are tacit? These are questions that need to be addressed in future research.

The amount of signifiers of algebraic objects level out between the grade 5 and grade 8 mathematics textbooks. Furthermore, the mean value of the signifiers length as well as the amount of signifiers of algebraic objects containing six words or more increase, indicating an overall greater complexity in the grade 8 mathematics textbook compared to the grade 5 mathematics textbook. Expressing signifiers of algebraic objects as written statements might have its limits. The more words constituting the signifier of an algebraic object, the more information it brings. Crowded in a single signifier of algebraic object, this information gets “entangled” and hard to interpret. Compared to symbols, words, are voluminous.
As described earlier, formal algebraic discourse sprung from a deliberate effort to reduce ambiguity and also to compress and standardize meta-arithmetic discourse. Symbolization became a means to this end, and made it possible to say more with less (Sfard, 2012). One way to approach the settling amount of signifiers of algebraic objects, denoting unknowns, is to parallel this development with that of history. That is, as discourse grows it comes to a point where words are replaced with symbols in order to say more with less. This might be what happens in the higher grades in compulsory school-books. Looking into this aspect of development in a research study would further broaden knowledge about elementary algebraic discourse evolvement.
9. References


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Licentiate dissertations in Pedagogical Work


